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E L E M E N T S
O F T H E
C O N I C S E C T I O N S,

By the late Dr ROBERT SIMSON, Professor of
Mathematics in the University of Glasgow. K

THE FIRST THREE BOOKS,

Translated from the LATIN original.

12 9/10
For the use of Students of Mathematics,

EDINBURGH:

Printed for CHARLES ELLIOT.

Sold, in London, by T. CADELL, and J. MURRAY.

MDCCLXXV. 19



Entered in Stationers Hall according to act
of parliament.

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The first three books of Dr Simon's treatise of the CONIC SECTIONS are translated into English, with a view to facilitate the study of the higher Geometry. These books contain as much of the doctrine as usually enters into an academical education : but if this specimen shall be found useful, and be honoured with the approbation of the Learned, the Public will be presented with a translation of the whole. A work composed by a Briton, and which is useful over all Europe, would seem to be a proper and valuable addition to the stock of English Literature.

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E L E M E N T S
OF THE
CONIC SECTIONS.

B O O K I.

Of the PARABOLA.

D E F I N I T I O N S.

- I. **A** Straight line AB, and C a point Fig. 1.
without it, are given in position. On the plane of AB, C
there is placed a ruler DEF, having its
side DE applied to AB, and its other side
EF on the same side of AB with the point
C. A string FGC is taken equal in length
to EF: and one end of this string being
fixed in F, and the other in C, a part of
A it

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it FG is, by means of a pin G, brought close to the side FE of the ruler; then, the string being kept uniformly tense by the pin, the side DE of the ruler is moved along AB: And thus the point of the pin, as it moves onwards with the ruler, describes upon the plane a line, named the **PARABOLA**. This line may be extended to a distance from the point C, exceeding any given distance, provided the length of the side FE of the ruler employed, be greater than that given distance.

II. The straight line AB is named the *directrix*.

III. And the point C is named the *focus* of the parabola.

IV. A straight line perpendicular to the directrix, is named a *diameter*; and the point where a diameter meets the parabola, is named the *vertex of that diameter*; and the diameter which passes through the focus, is named the *axis of the parabola*; and the vertex of the axis is named the *principal vertex*.

V. When a straight line terminated both ways by a parabola, is bisected by a diameter, it is said to be *ordinately applied*

to

to that diameter; or, it is named, simply, an *ordinate* to that diameter.

VI. A straight line quadruple of that segment of a diameter which is intercepted between its vertex and the directrix, is named the *latus rectum*, or the *parameter*, of that diameter.

VII. A straight line meeting a parabola only in one point, and which, when produced both ways, falls without the parabola, is said to *touch* the parabola in that point.

PROPOSITION I. THEOREM.

A straight line drawn perpendicular to the directrix from any point of a parabola, is equal to the straight line drawn to the focus from that same point.

Let G be a point in the parabola, and GE a straight line perpendicular to the directrix AB; draw GC to the focus C, and let EF be equal to the length of that side of the ruler which is on the same side of AB with the focus C: therefore EF is equal to the length of the string

Fig. 1.

A 2

FGC:

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FGC: take away the common part FG, and the remainder EG will be equal to the remainder GC.

COROLLARY. Hence that segment of the axis which is intercepted between the focus and the directrix, is bisected in the vertex of the axis. Thus CB is bisected in H.

PROP. II. THEOR.

If the distance of any point from the focus of a parabola be equal to the perpendicular drawn from that point to the directrix, that same point is in the parabola.

Fig. 1.
n. 2.

Let there be a parabola, the directrix of which is AB, and the focus C; and let D be a point, the distance of which from the focus is the straight line DC; from D draw DE perpendicular to the directrix. If DC be equal to DE, the point D is in the parabola.

From the center C, at the distance CD, describe a circle, meeting the axis in the point F: let H be the vertex of the axis,

I

and

and join CE: then, because any two sides of a triangle are together greater [20. 1. Elements of Euclid] than the third side, CD, DE are together greater than CE; much more, then, are they together [19. 1. Elem.] greater than CB: but CD is equal to DE, as also CH [cor. 1. 1.] to HB: therefore CD, that is, CF, is greater than CH: the parabola, therefore, with respect to its vertex H, is within the circle GDF: of consequence, it must meet the circle some where, since it may be extended [def. 1.] to a distance from the focus C which shall exceed any given distance. Now it meets the circle in the point D; for if this is not true, it must meet the circle in some other point. Let, then, the point L, which is on the same side of the axis with the point D, be that other point; then, having joined CL, draw LM perpendicular to the directrix, and LN parallel to the same; and let LN meet DE in N; and, because the point L is in the parabola, CL is [1. 1.] equal to LM; and, according to the hypothesis, CD is equal to DE; and, C being the center of the circle, CL is equal to CD: therefore LM, that is, NE, is equal to DE; which is impossible: the parabola, therefore,

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fore, meets not the circle in the point L, nor any where but in D: therefore D is a point in the parabola.

PROP. III. THEOR.

Any straight line drawn through the focus meets the parabola; and a straight line drawn from any point within a parabola to the focus, is less than the perpendicular drawn from that point to the directrix. A straight line, on the other hand; drawn from any point without a parabola to the focus, is greater than the perpendicular drawn from that point to the directrix.

Fig. 1. Let there be a parabola, the directrix of which is AB, and the point C the focus; any straight line drawn through C meets that parabola.

First, if CB, a straight line drawn through the focus, be perpendicular to the directrix, the point H, bisecting
[cor.

[cor. 1. 1.] the segment, intercepted between the focus and the directrix, is in the [2. 1.] parabola: but if any other straight line CP be drawn through the focus, bisect the angle BCP by the straight line CM, and let CM meet the directrix in M, and draw MN parallel to the axis BC: then, because the angles PCM, CMN are together less than two right angles, for each of them is less than one right angle, the straight lines CP, MN meet each other; let them meet in the point O, then the angle OCM is equal to the angle CMO; for each of the two is equal to [29. 1. Elem.] BCM; of consequence OM is [6. 1. Elem.] equal to OC: therefore the point O is in [2. 1.] the parabola.

To proceed to demonstrate the other part of the proposition: First, let there be any point K within the parabola, that is, let it be on the same side of it with the focus C, and draw KL at right angles to the directrix; draw likewise KC to the focus; KC is less than KL. Let CK meet the parabola in O, and let there be drawn to the directrix the straight line OM parallel to KL, and let OL be joined. Since, then, the point O is in the parabola, OC

is

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is equal to OM ; but OM is [19. 1. Elem.] less than OL ; much more therefore is $\{OM, \text{ or } OC\}$ less than [20. 1. Elem.] OK and KL together: take away the common part OK , and the remainder KC is less than the remainder KL .

Next, let the point Q be without the parabola, and draw QR at right angles to the directrix: QC drawn to the focus is greater than QR . Let QC meet the parabola in O , and draw OM parallel to QR , and join QM : therefore, because CO is equal to OM , CQ is equal to MO together with OQ ; but MO together with OQ , is greater than QM ; much more, then, are they greater than QR . QC is, therefore, greater than QR .

COR. Hence it is evident, that any point is within or without a parabola, according as the distance of that point from the focus is less or greater than a perpendicular drawn from that same point to the directrix.

PROP.

PROP. IV. THEOR.

A perpendicular to the directrix meets the parabola only in one point ; and when produced downwards, it falls within the parabola.

Let MT be perpendicular to the directrix AB ; draw MC to the focus; let CO be drawn, making the angle MCO equal to CMO , and meeting MT in O ; OM is consequently equal to OC ; and therefore the point O is in [2. 1.] the parabola. Fig. 1.

Next, take any point T in MO produced, and join TC : since, then, the angle MCT is greater than MCO , that is, than the angle CMT , TM is greater than TC : the point T therefore is within [cor. 3. 1.] the parabola. In like manner it may be demonstrated, that any point above O in the straight line MT is without the parabola.

B

PROP.

PROP. V. THEOR.

If from a point in a parabola a straight line be drawn to the focus, and if from the same point a straight line be drawn perpendicular to the directrix; the straight line which bisects the angle contained by these two straight lines, touches the parabola in that point. Also a straight line drawn through the vertex of the axis at right angles to the axis, touches the parabola.

Fig. 2. I, D being a point in a parabola, and DC drawn from D to the focus, and DA drawn perpendicular to the directrix AB; DE, that bisects the angle CDA, touches the parabola in the point D.

In DE take any other point F; and having joined FA, FC, AC, draw FG perpendicular to the directrix: then, because DA is equal [I. 1. Elem.] to DC, DF com-

mon, and the angle FDA equal to FDC; FC is equal [4. 1. Elem.] to FA; and, consequently, greater than FG: therefore the point F is without the [cor. 3. 1.] parabola; and, consequently, the straight line DE touches the parabola [def. 7.] in the point D.

2. HK, a straight line drawn through Fig. 2.
the vertex of the axis, and made perpendicular to the axis, touches the parabola. In HK take any point K; from which draw KL perpendicular to the directrix; and join KC: and, because KC is greater [19. 1. Elem.] than CH, that is, than [cor. 1. 1.] HB, that is, than KL, KC is greater than KE: therefore the point K is without the parabola, and HK touches the parabola.

COR. 1. This proposition points out a method of drawing a straight line that will touch a parabola in a given point, provided the directrix and the focus be given in position.

COR. 2. And since it has been proved, that all straight lines that touch a parabola, fall without it towards the same

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parts, that curve, it is plain, is every where convex on the side on which the touching lines are, but concave on the contrary side.

PROP. VI. PROBLEM.

The directrix and the focus of a parabola, and a straight line not parallel to any diameter, being given in position; to draw a straight line parallel to the straight line not parallel to any diameter, and which will touch the parabola.

Fig. 2. AB being the directrix and C the focus of a parabola, and MN a straight line not parallel to any diameter; it is required to draw a straight line parallel to MN, and which will touch the parabola.

From the focus C draw CO perpendicular to MN, and meeting the directrix A; and having bisected AC in E, draw ED parallel to MN, and meeting the diameter through A in the point D, and join CD; then,

then, in the triangles ADE, CDE, AE is equal to CE, ED common, and the angles at E right angles; DA, therefore, is equal to DC; and, consequently, the point D is in the [2. 1.] parabola: And, since the angle ADE is equal to the angle CDE, the straight line DE, as was shewn in the preceding proposition, touches the parabola.

PROP. VII. THEOR.

If from a point in a parabola, there be drawn a straight line, neither parallel to the axis, nor bisecting the angle contained by the diameter passing through that point, and a straight line drawn from the same point to the focus; that straight line cuts the parabola in one other point, but not in more than one.

If from E, a point in the parabola, the straight line EG be drawn, so as neither to be parallel to the axis, nor to bisect the angle

Fig. 3. 4.
n. 1. 2.

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angle contained by the diameter EF and the straight line EC ; from the focus C let a perpendicular be drawn to EG , and let it meet the directrix in A ; and, making Af equal to AF , through f draw fe parallel to FE , and let fe meet EG in e ; the point e will be in the parabola.

Fig. 3.

There are two cases. The one is that in which EG passes through the focus: Because EC is equal to EF , and each of the angles ECA , EFA a right angle; therefore AC is [5. and 6. 1. Elem.] equal to AF ; and, consequently, it is equal to Af ; and each of the angles ACe , Afe is a right angle: eC is, therefore, equal to ef ; and therefore the point e is in the [2. 1.] parabola.

Fig. 4.
n. 1. 2.

In the other case, EG passes not through the focus. From the centre E , at the distance EC , describe a circle, meeting CA again in H ; and describe another circle through the points C , H , f ; then, because EC is equal to EF , and that EFA is a right angle, the circle described from the centre E touches [16. 3. Elem.] the directrix in F : therefore the rectangle CAH is equal to [36. 3. Elem.] the square of AF , that is, to the square of Af : therefore Af touches the circle

circle [37. 3. Elem.] fCH ; and the centre of this circle is [19. 3. Elem.] in fe ; it is also in GE , which bisects CH at right angles: it is, therefore, in the point e where fe , GE intersect each other. eC , therefore, is equal to ef ; and, therefore, the point e is in the [2. 1.] parabola.

It is evident, that EG cuts the parabola no where but in the points E , e : For, if possible, let EG cut it also in another point ϵ ; and let $\epsilon\phi$ be drawn perpendicular to the directrix AB ; a circle, then, described from the centre ϵ , distance ϵC , passes through H , and touches the directrix in the point ϕ , at a distance from the point A , less or greater than that of the point F or f from A ; and the square of ϕA being equal to the rectangle CAH , is equal to the square of FA : which is absurd.

Fig. 4.
n. 1. 2.

n. 1. 2.

COR. Of all the straight lines that can be drawn from any point of a parabola, only one can touch the parabola; for the diameter through the point falls [4. 1.] within the parabola; and any other straight line, except that which bisects the angle contained by the diameter through the point, and the straight line

line drawn from the point to the focus, meets the parabola again in another point.

PROP. VIII. THEOR.

If from the focus of a parabola to a straight line there is drawn a perpendicular meeting the directrix; if the segment of the perpendicular intercepted between the focus and the straight line, is not greater than its other segment intercepted between the straight line and the directrix; that straight line, necessarily, meets the parabola.

Fig. 4
n. 1. 2.

Let C be the focus of a parabola, and let CG, which meets the directrix in A, be a perpendicular to the straight line LG; if the segment CG, between C and LG, be not greater than the segment GA, between LG and the directrix, the straight line LG necessarily meets the parabola.

When the segments CG, GA are equal, it is plain, from what was demonstrated in

in Prop. 6. that the straight line LG touches the parabola in the point where the diameter through A meets the same LG.

But if CG be less than GA, take GH equal to GC; and from the point A, and on either side of A, place, in the directrix, AF, or Af, such, that the square of either may be equal to the rectangle CAH; and having described a circle through the points C, H, F, draw, through F, FE perpendicular to AF, and let FE meet LG in E: and the square of AF being equal to the rectangle CAH, AF touches the circle in F; and therefore the centre of the circle is in FE: but as CH is bisected at right angles by the straight line LG, the centre of the circle is likewise in LG: it is, therefore, in E, the point where FE, LG intersect each other: hence EC is equal to EF; and, of consequence, the point E is in the parabola. In like manner, if *ef* drawn perpendicular to the directrix meets LG in *e*, the point *e* is in the parabola.

COR. Hence any straight line passing through a point within a parabola, meets the parabola.

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Case 1. If the straight line is a diameter, it is evident, from Prop. 4. 1. that it meets the parabola.

Case 2. When the straight line is not a diameter. Let LG pass through the point L within the parabola; it will meet the parabola. From the focus C let the straight line CG be drawn at right angles to LG, and let it meet the directrix in A; and join LC, LA: then because the point L is within the parabola; a straight line drawn from L perpendicular to the directrix, is greater [3. 1.] than LC: LA, therefore, which is not less than this perpendicular, is greater than LC: AG, therefore, is greater than [47. 1. Elem.] GC; and therefore LG meets the parabola.

PROP. IX.

The angle contained by a diameter of a parabola, and a straight line drawn from the vertex of that diameter to the focus, is bisected by the straight line that touches the parabola in that vertex,

The

The angle ADC , contained by the diameter AD , and the straight line DC , is bisected by DE , a straight line touching the parabola in the vertex D : for if DE bisects not the angle ADC , it is possible for some other straight line to do it; and this other straight line also will touch the parabola: which is absurd. Fig. 3.

COR. 1. On the other hand, if AD be a diameter, and ED touch the parabola in the vertex D of AD , and if the angle ADE be equal to the angle CDE , DC passes through the focus: Or, if DC passes through the focus, DE touching the parabola in D , and the angle ADE being equal to the angle CDE ; DA is a diameter.

COR. 2. If from any point D which is in a parabola, but which is not the vertex of the axis, a straight line DE be drawn touching the parabola, the angle EDA contained towards the directrix by the straight line DE and the diameter DA , is less than a right angle; for the angle ADC , which is the double of EDA , is less than two right angles.

PROP. XX. THEOR.

If from a point in a parabola a straight line be drawn touching the parabola, and if from the same point a perpendicular be drawn to the axis; the segment of the axis intercepted between the perpendicular and the touching line, is bisected in the vertex of the axis.

Fig. 5.

Let D be a point of a parabola, and let DE drawn from D touch the parabola, and DH be perpendicular to the axis; the segment EH of the axis is bisected in F , the vertex of the axis.

Through D let DA be drawn perpendicular to the directrix; let DC be drawn to the focus, and let the axis meet the directrix in B : And because the angle CDE is equal to the angle ADE , that is, to the alternate angle CED , CE is equal to CD , or DA , that is, to HB ; and CF is equal to FB ; therefore the remainder FE is equal to the remainder FH .

PROP.

PROP. XL. THEOR.

Every straight line parallel to a straight line that touches the parabola, and terminated both ways by the parabola, is bisected by the diameter passing through the point of contact, that is, it is ordinately applied to this diameter.

The straight line Ee , which is terminated in the points E, e , being parallel to DK , a straight line touching the parabola; and AD , the diameter which passes through the point of contact D , meeting Ee in L ; LE is equal to Le .

Fig. 4.
n. 1. 2.

Let AD meet the directrix AB in A ; from the points E, e to the directrix, draw the perpendiculars EF, ef ; and from the focus C draw CA meeting Ee in G ; and from the centre E , distance EC , describe a circle meeting CA again in H ; this circle will touch the directrix in F : join DC : then, because DA is equal to DC , and the angle ADK equal to $[9. 1.] CDK$, DK is $[4. 1. Elem.]$ perpendicular to AC : and, therefore,

therefore, Ee too is at right angles to the same AC : and because E is the centre of the circle CFH , CG is equal to $[3. 3. \text{Elem.}]$ GH : join eC and eH , and eC will be equal $[4. 1. \text{Elem.}]$ to eH : A circle, therefore, described from the centre e , and at the distance eC , passes through H ; and eC being equal to ef , it passes likewise through f : therefore, since the straight line Ff touches the circles, and the straight line AHC cuts them, the square of AF is equal to the $[36. 3. \text{Elem.}]$ rectangle CAH , that is, to the square of Af : therefore AF is equal to Af ; and FE , Al , fe , are parallels: therefore LE is $[2. 6. \text{Elem.}]$ equal to Le .

COR. I. On the other hand, if a straight line Ee , terminated both ways by a parabola, be bisected by the diameter AL , it is parallel to the tangent which passes through D , the vertex of AL : for if the straight line touching the parabola in the point D , be not parallel to LE , let another straight line be drawn touching the parabola, and parallel to LE ; then the diameter which passes through the point where this other straight line touches the parabola,

la, bisects the straight line Ee : but, according to the hypothesis, the same Ee is bisected by the diameter AL : which is absurd.

COR. 2. All straight lines ordinately applied to any diameter, are parallel to one another.

COR. 3. If two or more parallels be terminated both ways by a parabola, the diameter which bisects the one, or one of them, bisects also the other, or the rest of them: for the one that is bisected by a diameter, is parallel to the straight line touching the parabola in the vertex of that diameter; and consequently the other, or the others, is, or are, parallel to the same straight line that touches the parabola in that vertex; and, therefore, is, or are, bisected by the same diameter.

COR. 4. Any straight line, on the contrary, which bisects two parallels terminated both ways by a parabola, is a diameter: for if it is not, it is possible for some other straight line bisecting one of the parallels to be a diameter; and being a diameter, this other straight line must also bisect the other of them: but, according to the hypothesis, the former of
the

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the straight lines bisects both the parallels : which is absurd. And if from the point of contact a straight line be drawn bisecting another straight line parallel to the tangent, and terminated both ways by the parabola, that straight line is a diameter : for if it be not, let a diameter be drawn through the point of contact, this diameter must also bisect the parallel to the tangent : which is absurd.

COR. 5. And a straight line drawn through the vertex of a diameter, so as to be parallel to straight lines ordinately applied to that diameter, touches the parabola. This is manifest from cor. 1.

PROP. XII. THEOR.

If from a point of a parabola a straight line be drawn perpendicular to a diameter, and if from the same point a straight line be ordinately applied to that diameter ; the square of the perpendicular is equal to the rectangle contained by the abscissa of the diameter

diameter and the *latus rectum* of the axis.

[N. B. An *abscissa* is the segment intercepted betwixt the vertex of a diameter and a straight line ordinately applied to that diameter.]

Case 1. When the diameter is the axis of the parabola.

Let D be a point in a parabola, and DH Fig. 5.
a perpendicular to the axis BC; DH will be parallel to [5. 1.] the straight line touching the parabola in the vertex of the axis; and therefore will be ordinately [11. 1.] applied to the axis: draw DC to the focus, and DA perpendicular to the directrix AB, and let F be the vertex of the axis: then, because HB is equal to DA, that is, to DC, the square of HB is equal to the square of DC, that is, to the square of DH, together with the square of HC: but, since BF is equal to FC, the same square of HB is equal to four times the rectangle HFC, together with the [8. 2. Elem.] square of HC: therefore the square of DH, together with the square of HC, is equal to four times the rectangle

D HFB,

HFB, together with the square of HC; therefore the square of DH is equal to four times the rectangle HFB, that is, to the rectangle contained by the abscissa HF, and the parameter of the axis.

Case 2. When the diameter to which the perpendicular is drawn is not the axis.

Fig. 4.
A. 1. 2.

Let EN be perpendicular to the diameter AD; let EL be an ordinate to AD, and D the vertex of the same AD; the square of EN is equal to the rectangle contained by the abscissa LD and the parameter of the axis.

Draw DK parallel to LE; DK will therefore [5 cor. 11. 1.] touch the parabola in D; and let the same DK meet the axis in K; let EF be drawn at right angles to the directrix; and let a circle be described from the centre E, distance EF; and this circle will touch [cor. 16. 3. Elem.] the directrix in F, and pass through the focus C: let AC be joined, and let it meet the circumference of the circle again in H, and the straight lines DK, LE in the points P, G; and let LE meet the axis in O.

Because the angles [9. 1. of this book, 4. 1. Elem.] CPK and CBA are right angles,
and

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those straight lines and the vertex of that diameter. Let EL , QR be ordinately applied to the diameter DN ; and let EN , QS be perpendicular to the same: because the triangle ELN is equiangular to the triangle QRS , the square of EL is to that of QR , as the square of EN to that of QS , that is, by the preceding corollary, as the abscissa DL to the abscissa DR .

COR. 3. And if from the vertices of two diameters there be drawn straight lines ordinately applied to those two diameters, that is, if the straight line drawn from the vertex of each diameter be an ordinate to the other diameter, the abscissas between those ordinates and the two vertices are equal to each other; for the perpendiculars drawn from the two vertices to the two diameters are equal.

P R O P. XIII. THEOR.

If from a point of a parabola a straight line be drawn ordinately applied to a diameter, the square of half that ordinate is equal to the rectangle contained by the abscissa

abscissa between that same ordinate and the vertex of that diameter, and the *latus rectum* of the same diameter.

Let AB be the directrix of a parabola, and AD a diameter, to which EL, drawn from the point E of the parabola, is ordinately applied; and through the vertex D of the diameter AD, draw DK parallel to EL; DK, of consequence, will touch the parabola: draw DM perpendicular to the axis, and from Q, the vertex of the axis, draw QR ordinately applied to the diameter DL; and, consequently, parallel to EL.

Fig. 4.
n. 1. 2.

Since QR is equal to DK, its square is equal to [47. 1. Elem.] the squares of DM, MK; but the square of DM is, by the first case of the foregoing proposition, equal to four times the rectangle MQB: and since MQ is equal [10. 1.] to QK, the square of MK is equal to four times the square of MQ: therefore the square of QR is equal to four times the rectangle MQB, together with four times the square of MQ, that is,

to

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to four times [3. 2. Elem.] the rectangle QMB: but MQ, or QK, is equal to DR, and MB to DA; therefore the square of QR is equal to four times the rectangle RDA: and since QR, EL are ordinately applied to the diameter AD, the square of QR is to the square of EL as [2 cor. of the preceding proposition] RD to LD, that is, as four times the rectangle RDA to four times the rectangle LDA: but the square of QR, as hath been proved, is equal to four times the rectangle RDA; therefore the square of EL is equal to four times the rectangle LDA, that is, to the rectangle contained by the abscissa LD and the parameter of the diameter AD.

It was from the property above demonstrated, that Apollonius named the curve line which is the subject of this book the PARABOLA.

COR. I. If from a point E to AD, a diameter of the parabola, a straight line EL is drawn parallel to straight lines ordinately applied to the diameter AD, and meeting the same AD below its vertex D; if the square of EL is equal to the rectangle contained by the abscissa LD and the parameter

parameter of the diameter AD; the point E is in the parabola.

For since the point L is within [4. 1.] the parabola, the straight line EL necessarily meets [cor. 8. 1.] the parabola: if, therefore, EL does not meet the parabola in the point E, on the same side of the diameter with E, let it meet it, if possible, in some other point, nearer, or more remote, from the diameter, than E is: let this other point be ϵ ; then the square of ϵL is equal to the rectangle contained by LD, and the parameter of the diameter, that is, according to the hypothesis, to the square of EL: which is absurd.

Fig. 4.
n. 2.

COR. 2. If from two points E, Q, one of which, Q, is in the parabola, there be drawn to the diameter AD straight lines EL, QR, parallel to straight lines ordinately applied to AD; if the squares of the parallels be to one another as the abscissas between the parallels and the vertex of the same AD; the other point E is also in the parabola.

If the diameter LD meets the directrix in A, four times AD is its *latus rectum*: then, since the square of QR is to that of EL as RD is to LD, that is, as four times the

the rectangle RDA to four times the rectangle LDA; and since, by the proposition, the square of QR is equal to four times the rectangle RDA; the square of EL is equal to four times the rectangle LDA; and therefore, by the preceding corollary, the point E is in the parabola.

PROP. XIV. THEOR.

If a straight line be drawn from a point of a parabola so as to be ordinately applied to a diameter, and if another straight line be drawn from the same point so as to touch the parabola, and meet that diameter; the segment (of the diameter) intercepted betwixt the ordinate and the tangent is bisected in the vertex of the diameter.

Fig. 6. A being a point of a parabola; AC drawn from A, so as to be ordinately applied to the diameter BC, and AD drawn from the same point, so as to touch the parabola,

parabola, and meet BC in D; the segment CD is bisected in B, the vertex of BC.

From the vertex B let BE be drawn parallel to AD; it will be ordinately [11. 1.] applied to the diameter AE, and the abscissa BC will be equal to the abscissa [3. cor. 12. 1.] AE, that is, to BD.

COR. 1. On the other hand, AC being ordinately applied to the diameter BC, if AD be drawn making BD equal to BC, AD touches the parabola.

For if AD does not touch the parabola, let AF touch it; FB then is equal to BC: which is impossible.

COR. 2. If a straight line touches a parabola, its segment between the point of contact and any diameter is bisected by a straight line touching the parabola in the vertex of that diameter.

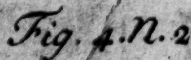
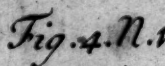
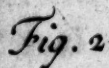
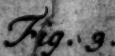
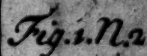
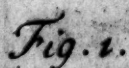
Let AD, a straight line touching the parabola, meet the diameter CB in the point D, and the tangent GB in G; let AC be drawn parallel to BG; AC will be ordinately applied to the [11. 1.] diameter CB: and since, by the proposition, CB is equal to BD, AG is likewise equal to GD.

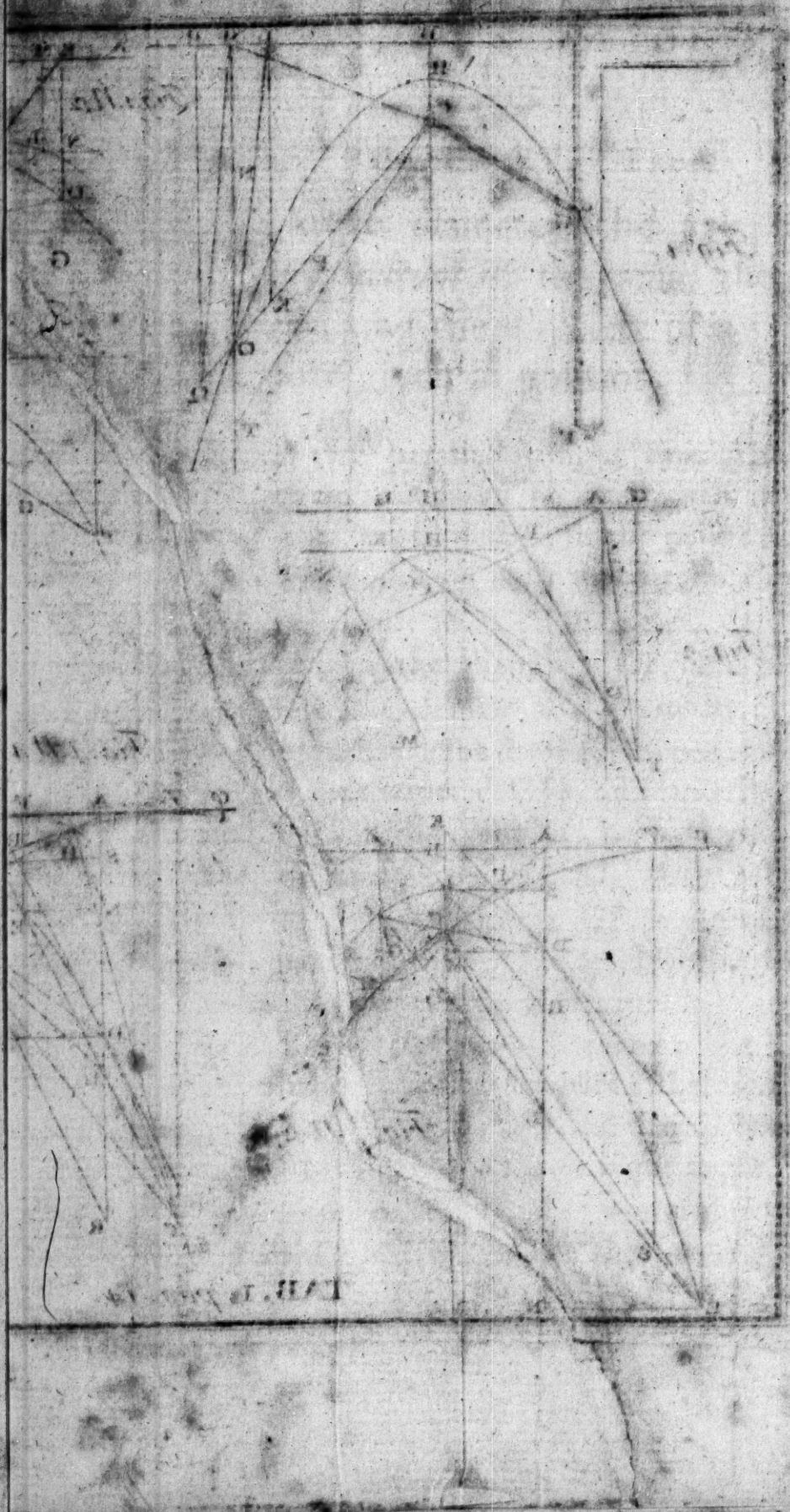
PROP. XV. PROB.

To find a diameter, the axis, the *latus rectum* of the axis, the focus, and the directrix of a parabola given in position.

Fig. 7. Let two parallel straight lines AB, CD be drawn; let them be terminated in the parabola, and bisected in the points F, E; join FE, and let it meet the parabola in G: GF is [4. cor. 11. 1.] a diameter.

Next, in the diameter GF, and below its vertex G, take any point H; and through that point draw KHL perpendicular to the diameter GF, and meeting the parabola in the points K, L; and through M, the middle point of KL, draw MN parallel to the diameter GF, and meeting the parabola in N; and let NO be drawn parallel to MH: then, because MN is parallel to GH, it is a diameter: but KL is ordinately applied to MN; therefore NO touches [5. cor. 11. 1.] the parabola in N. And because MNO is a right angle, MN is [2. cor. 9. 1.] the axis: and a third proportional to NM, MK is the [13. 1.] *latus rectum*





rectum of the axis: and the distance of the focus from the vertex of the axis is equal to a fourth part of the *latus rectum* of the axis; therefore the focus is given. After the same manner is the directrix found.

PROP. XVI. PROB.

The directrix and the focus of a parabola being given in position, to describe the parabola.

Let AB be the directrix, and C the focus, and with a ruler and string describe the [def. 1.] parabola; or as many points of the parabola as may be thought necessary may be thus found; through the focus C draw CB at right angles to the directrix, and CB will be the axis: to the axis CB draw any perpendicular LG, meeting it below its vertex F, and in the same axis place CH equal to BG; CH will thus be greater than CG; and from the centre C, distance CH, describe a circle, cutting the straight line LG in D, *d*; these points are in the parabola.

Fig. 8.

For the straight line DA, drawn to the directrix, so as to be parallel to CB, is e-

qual to GB, that is, to CH, that is, to CD; and D, therefore, is [2. 1.] in the parabola. In the same manner it may be shewn, that d is in the parabola.

COR. Hence, if the directrix AB of a parabola, and F, the vertex of the axis, be given in position, the parabola may be described, by drawing FB at right angles to the directrix, and making FC equal to FB; for C will be the focus. In like manner, if the vertex F and focus C be given; join CF, and produce it to B, so that FB may be equal to FC; a straight line drawn through B at right angles to BC, will be the directrix: and if the axis GF, its vertex F, and its parameter FK, be given in position, the directrix may be found by making FB equal to a fourth part of the parameter FK, and drawing BA at right angles to the same FB. In like manner the directrix may be found, if the axis, the focus, and the parameter of the axis, be given in position. In all these cases, therefore, the parabola may be described according to the proposition.

P R O P.

PROP. XVII. PROB.

The axis and its vertex, and a point without the axis and below its vertex, being given in position, to describe the parabola which will pass through that point.

The axis FH , its vertex F , and D a point without it, and below the vertex F , being given in position; it is proposed to describe the parabola which shall pass through D . Fig. 3.

Having drawn from the point D , GD perpendicular to the axis, find [11. 6. Elem.] FK a third proportional to the two straight lines FG , GD ; then, taking FB equal to the fourth part of it, and making FC equal to FB , draw BA parallel to DG ; and let a parabola be described, having C for its focus, and AB for the directrix; this parabola will pass through the point D . For since FG , GD , FK are proportionals, the square of GD is equal to the rectangle GFK : and FK is the parameter of

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of the [def. 6. 1.] diameter FG ; therefore the point D [1. cor. 13. 1.] is in the parabola.

PROP. XVIII. PROB.

Fig. 9. Two straight lines AB, AC, which meet each other in the point A, being given in position, and a straight line DE being given in magnitude ; to describe a parabola which may have AB for a diameter, and DE for the parameter of AB, and which the straight line AC may touch in the point A.

Take of DE the fourth part DF, and in BA produced place AG equal to DF, and draw GH at right angles to AG ; then, making the angle CAK equal to GAC, and the straight line AK equal to AG, describe a parabola, which may have K for the focus, and GH for the directrix ; AB will be one of the diameters, and the parameter of AB will be equal to the

[def.

[def. 4. 6.] quadruple of AG , that is, to DE : and since AG is equal to AK , the point A is in the parabola; and since the angle GAC is equal to CAK , the straight line AC touches the parabola in the point A .

COR. If from a point L , a straight line LM be drawn in a given angle LMA , to a straight line AB given in position; and if the square of LM be equal to the rectangle contained by a given straight line DE , and the segment MA , intercepted between the same LM and the given point A ; the point L is in a parabola given in position. Having drawn through the point A a straight line AC parallel to LM , describe, according to the proposition, a parabola which may have AB for a diameter, and DE for the parameter of AB , and which the straight line AC may touch in the point A ; this parabola is the *locus* of the points L ; for since the square of LM is, by hypothesis, equal to the rectangle contained by MA and DE , and that LM is parallel to AC that touches the parabola, and consequently to straight lines ordinately applied to the diameter MA ; the point L is in the [1. cor. 13. 1.] parabola.

P R O P,

PROP. XIX. PROB.

Fig. 9. A diameter AB , and its vertex A , being given in position, and a straight line LM , which meets AB in M below the vertex, being given in position and magnitude; to describe a parabola which may pass through the point L , and in which the straight line LM may be ordinately applied to the diameter AB ,

Through the vertex A draw AC parallel to LM , and let DE be a third proportional to AM , ML ; and, according to the preceding proposition, describe a parabola which may have AB for a diameter, and DE for the parameter of AB , and which AC may touch in the point A : then, because ML is parallel to AC , it is ordinately applied to the diameter AB ; and because AM , ML , DE are proportionals, the square of ML is equal to the rectangle contained

contained by the abscissa AM, and DE the parameter of the diameter AB; and therefore the point L is [1. cor. 13. 1.] in the parabola.

PROP. XX. PROB.

A diameter of a parabola, and the vertex of that diameter, being given in position, and the *latus rectum* of the same diameter being given in magnitude, and a point in the parabola being given; to describe the parabola.

Let AB be the diameter given in position, and A its vertex; in AB, and above the vertex A, place the straight line AC equal to the given *latus rectum*; and let D be the given point in the parabola. Suppose what is required done; and let AD be the parabola to be described: and having drawn the straight line AE touching it in A, and meeting the diameter drawn through D in the point E, complete the parallelogram AEDF: therefore DF is ordinately applied to the diameter AB; and therefore the square of DF, or AE, is equal to the rectangle FAC: as, therefore,

F

FA

Fig. 10.

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FA or DE to AE, so is AE to AC; and they contain the equal [29. 1. Elem.] angles DEA, EAC; therefore the triangle DEA is equiangular to the triangle EAC: and thus the angle AEC is equal to the angle EDA, or FAD. If, then, upon AC a segment of a circle be described, containing an angle equal to FAD, the point E will [converse of 21. 3. Elem.] be in the circumference of this segment. But the angle FAD is given, because FA, DA are [26. dat.] given in position; therefore the angle AEC is given: and AC is given in position and magnitude; therefore the segment AEC [8. def. dat.] is given in position. The point E, then, is in the circumference of a circle given in position; but it is also in the straight line DE which is given in position; the point E, therefore, is given: and the point A is given: therefore the straight line AE is given in position. It is possible, therefore, to describe [18. 1.] a parabola which may have AB for a diameter, and AC for the *latus rectum* of AB, and which AE may touch in the point A.

In order to the composition, it is required, that a segment of, a circle containing an angle equal to FAD, be described

bed upon AC, and that a straight line drawn through the point D, parallel to AB, meet the circumference of that segment. But these conditions it is sometimes impossible to fulfil. Hence the problem cannot always be solved.

When the straight line drawn through D parallel to AB, is a tangent to the segment, the problem admits of only one solution*. The parabola, and *latus rectum* of the diameter AB, which solve this case, are determined, if a parabola be found having AB for a diameter, and A for the vertex of AB, and which passes through the point D; and if the *latus rectum* of BA be such, that a segment of a circle described upon it, when placed above A, and in the direction of AB, may contain an angle equal to BAD; and that a straight line drawn through D, so as to be parallel to AB, may touch the circumference of that segment: suppose what is required done: let AG be the parameter of the diameter AB; and upon AG let the segment of a circle containing an angle equal to

* In all cases in which the straight line drawn through D parallel to AB cuts the circumference of the segment in two points, the problem admits of two solutions.

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the angle BAD, or ADE, be described; and let the straight line DE parallel to AB touch the circumference of that segment in H, and join AH, GH: since, then, the angle AHD is equal [32. 3. Elem.] to AGH in the opposite segment, and that, according to the hypothesis, the angle ADH is equal to AHG; the triangles ADH, AHG are equiangular; the angle DAH is, therefore, equal to HAG: but the angle DAG is given; and, consequently, its half DAH is given: and the straight line AD is given in position; therefore AH also is [29. dat.] given in position; the point H too is given, where AH meets DE, given in position; and the angle AHG is given: hence HG is given [29. dat.] in position; and therefore the point G is given: hence the straight line AG is given in magnitude: since, then, AG in the parabola, which passes through the point D, is the *latus rectum* of the diameter AB, of which A is the vertex; since a straight line touching the parabola in the vertex of the diameter AB, meets, as hath been proved, the diameter drawn through D, in the point where this diameter meets the circumference of the circle, the segment
of

of which, described upon the *latus rectum* of the diameter, passing through A, contains an angle equal to ADH; and since, in the present case, the diameter DE meets the circumference of this segment in H; HA touches the parabola in A: and AB, AH being given in position, and AG given in magnitude, the parabola, according to the 18th proposition, can be described.

The composition of this case is as follows: Join AD, and through D draw DE parallel to AB; to DE draw AH, bisecting the angle DAG; and through H to AB draw HG, making the angle AHG equal to ADH or DAB; and let a parabola be described which may have AB for a diameter, and AG for the *latus rectum* of AB; and which AH may [18. 1.] touch in A: this parabola will pass through D, and DH will touch the circle described about AHG. Draw DK parallel to AH; and since the triangles DAH, AHG are isosceles and equiangular, DH, HA, AG, and consequently KA, KD, AG, are proportionals; the square of DK is, therefore, equal to the rectangle KAG; and DK is parallel to the tangent AH: hence the
point

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point D is in the [1 cor. 13. 1.] parabola: and because the angle AHD is equal to AGH in the opposite segment, DH touches [conv. 32. 3. Elem.] the circle in H.

It remains to be inquired, whether the parameter AG be greater or less than the parameter of the diameter AB in any other parabola, having AB for a diameter, and A for the vertex of AB, and which passes through D. Let there be any other parabola admitting of these conditions; and let AE touch it in A, and meet the diameter passing through D in the point E, and the circle GHA in L: having joined LG, draw EC parallel to it; draw also DF parallel to EA; DF, therefore, is ordinately applied to the diameter AB: and because the angle ADE is equal to the angle AHG or ALG, that is, to AEC, and that the angle DEA is equal to EAC, the triangle EDA is equiangular to the triangle AEC: therefore the straight lines DE, EA, AC, that is, AF, FD, AC are proportionals; the square of DF is, therefore, equal to the rectangle FAC: and, for this reason, AC is the parameter of the diameter AB in this parabola. And because DE touches the circle in H, AL is less than AE; and
therefore

therefore AG is less than AC: therefore AG is the least of all the possible parameters of the diameter AB, in parabolas which have AB for a diameter, and A for the vertex of AB, and which pass through D. After the same manner it may be shewn, that in any parabola whatever, which answers the conditions of the proposition, the *latus rectum* of the diameter AB is greater or less, according as the tangent drawn through A, and situated on either side of the tangent AH, is more remote from, or nearer to, the same AH.

To proceed to the composition of what was analysed in the first case: If the proposed parameter be equal to AG, found in the manner above mentioned, the parabola, as hath been shewn, may be described, and will be the only one that can fulfil what is required in the problem. If, next, the proposed parameter be less than AG, it is impossible to construct the problem: or if the proposed parameter, for example, AC, be greater than AG; upon AC describe the segment of a circle containing an angle equal to ADH, or DAB; and since DH touches the circle AHG, it must cut the segment described upon AC
in

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in two points : let E be one of them ; and join AE ; and let a parabola [18. 1.] be described, having AB for a diameter, and AC for the parameter of AB, and to which the straight line AE may be a tangent ; and draw DF parallel to AE ; then it may be shewn, as above, that DE, EA, AC, that is, that AF, FD, AC, are proportionals ; that the square of DF is, therefore, equal to the rectangle FAC, contained by the abscissa FA and the parameter AC ; and that, consequently, the parabola passes through the point D. The same thing may be demonstrated with regard to the other parabola, which has for a tangent the straight line joining A, and the other point of intersection *e*. And as, in the investigation of the problem, it has been proved, that the angles GAH, HAD are equal ; the angle AKD is, therefore, equal to ADK ; and, of consequence, AK is equal to AD : but AG is a third proportional to AK, KD, or to AD, DK ; that is, the least parameter is a third proportional to the straight line which joins the vertex of the diameter given in position and the given point ; and the straight line which is drawn from the same point to the

the

the diameter, so as to cut off from the diameter a segment equal to the first proportional.

The first nine definitions in the first book of Apollonius of Perga's Conic Sections.

1. VIII. If a straight line joining any point and the circumference of a circle not in the same plane with the point, be produced from the point in the opposite direction, and then, while the point remains fixed, be carried round in the direction of that circumference till it return to the place from which the motion has commenced; by the revolution of that straight line, a surface, named *the conical surface*, and which consists of two surfaces connected together at the fixed point, will be described. The two connected surfaces may each of them be infinitely increased, if the straight line with which they are described be produced both ways to an infinite distance.

2. IX. The fixed point is named *the vertex* of the conical surface.

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3. X. The straight line drawn through the point and the centre of the circle is named *the axis*.

4. XI. The figure contained by the circle, and the surface which is intercepted between the vertex and the circumference of the circle, is named *the cone*.

5. XII. The same fixed point, which is the vertex of the surface of the cone, is named *the vertex of the cone*.

6. XIII. The straight line drawn from the vertex to the centre of the circle, is named *the axis of the cone*.

7. XIV. And the circle itself is named *the base of the cone*.

8. XV. Cones which have their axis at right angles to the base, are named *right-angled cones*.

9. XVI. And cones which have not their axis at right angles to the base, are named *scalene cones*.

PROP. XXI. [*Prop. 1. B. 1. Apoll.*]

Straight lines drawn from the vertex of the surface of a cone to points in that surface, are in that same surface.

Let

Fig. 5.

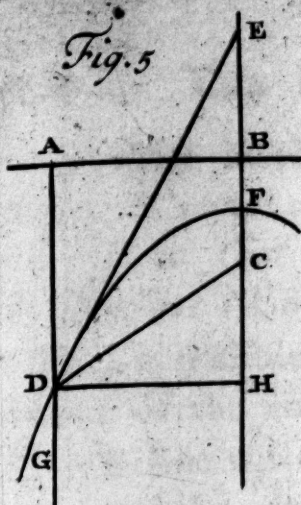


Fig. 6.

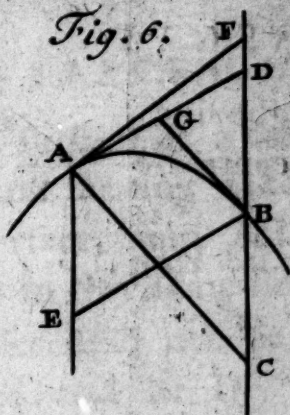


Fig. 7.

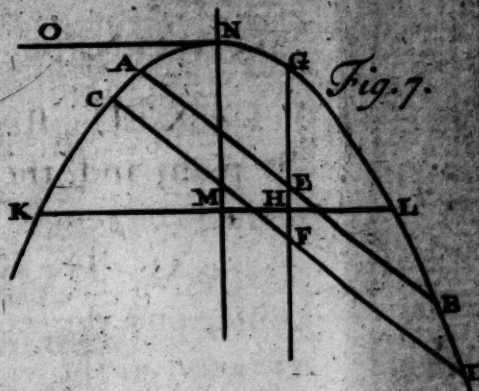


Fig. 8.

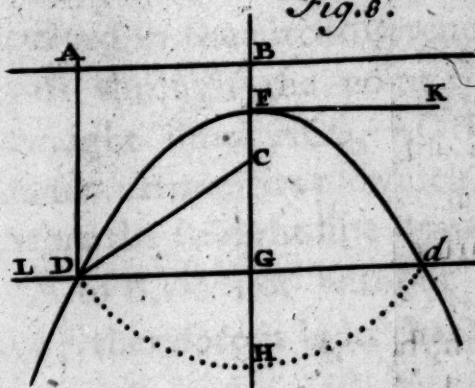


Fig. 10.

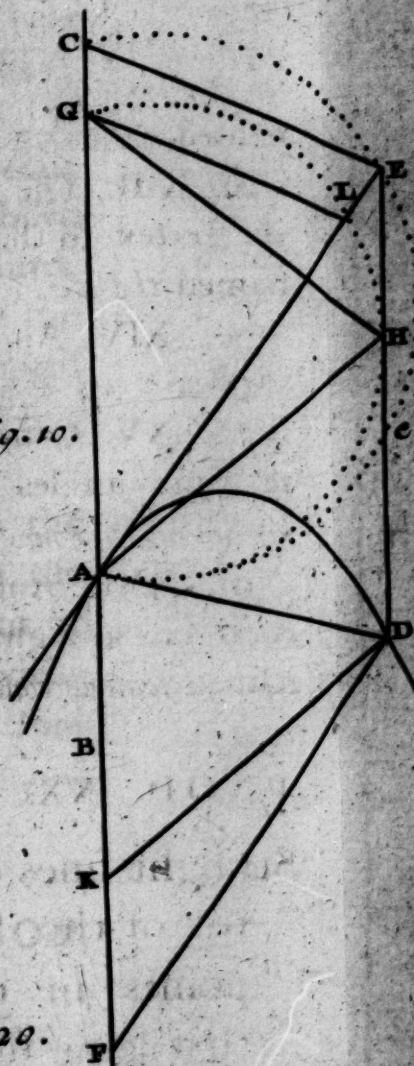
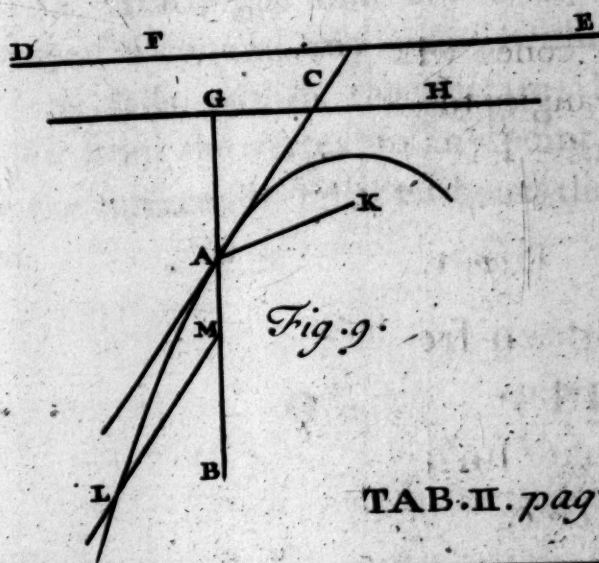
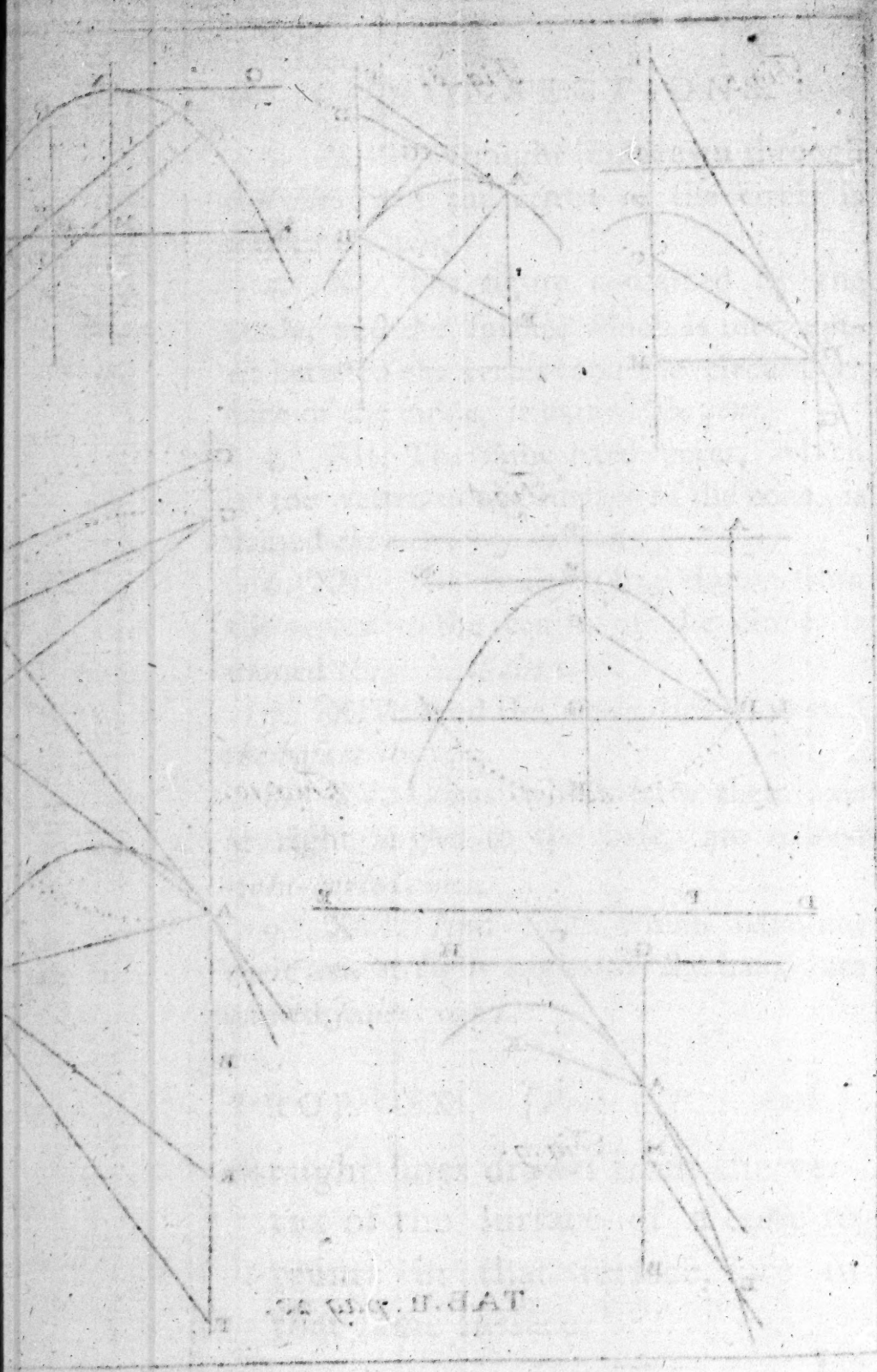


Fig. 9.





TAB. II. 2. 10. 20.

Let there be the surface of a cone: let A be its vertex; and having taken any point B in that surface, join ACB: the straight line ACB is in that same surface.

For, if possible, let ACB not be in the surface of the cone; and let DE be the straight line with which the cone is described, and the circle EF the base; and if DE be revolved in the circumference of EF, it will pass through the point B; and thus two straight lines ACB, AGB will have the same extremities: which is absurd. Therefore the straight line drawn from the point A to B, is not without the conical surface; therefore it is in that surface.

COR. A straight line drawn from the vertex of a cone to any point within the surface, falls within the surface; but if drawn from the vertex to any point without the surface, it falls without the surface.

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P R O P. XXH. [*Prop. 3. B. 1. Apoll.*]

If a cone be cut by a plane passing through its vertex, the section is a triangle.

Fig. 12.

Let there be a cone which has the point A for its vertex, and the circle BDC for its base; let it be cut through the point A by any plane; and let the sections made in the surface be the lines AB, AC, and the section in the base the straight [3. 11. Elem.] line BC; ABC is a triangle.

For since the straight line drawn from the point A to B, is both in the cutting plane and in [21. 1.] the conical surface, it is the common section of the two; therefore the section AB is a straight line: for a like reason, the section AC is a straight line; and BC too is a straight line: therefore ABC is a triangle.

P R O P. XXIII. [*Prop. 4. B. 1. Apoll.*]

If either of the surfaces at the vertex be cut by a plane parallel to the
the

the circle in the circumference of which the straight line that describes the conical surface is revolved; the plane inclosed within the surface is a circle having its centre in the axis; and the figure contained by this circle, and that part of the conical surface which is intercepted between the cutting plane and the vertex, is a cone.

Let there be a conical surface the vertex of which is A, BC being the circle in the circumference of which the straight line which describes the *surface* is revolved; let it be cut by any plane parallel to the circle BC, and let this plane make in it a section DLE: the line DLE is the circumference of a circle the centre of which is in the axis. Take the centre of the circle BC, and let it be F; join AF; AF, consequently, is [def. 10.] the axis, and meets the cutting plane; let it meet it in G; next, let any plane pass through the same AF; and the section made by this plane

Fig. 13.

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plane will be [22. 1.] a triangle ABC. And because the points D, G, E are both in the cutting plane DLE, and in the plane ABC, DGE is [3. 11. Elem.] a straight line. Again, in the line DLE take any point H; join AH, and produce it; AH then will meet the circumference BC; let it meet it in K, and join GH, FK: And because the two parallel planes DLE, BC are cut by the plane ABC, their [16. 11. Elem.] common sections with it are parallels: DE, consequently, is parallel to BC: and, for the same reason, GH is parallel to FK: therefore, as AF to [4. 6. Elem.] AG, so is FB to GD, FC to GE, and FK to GH; and the three straight lines BF, KF, CF, are equal; therefore the three straight lines DG, GH, GE [14. 5. Elem.] are also equal. After the same manner it may be demonstrated, that any other straight lines whatever, drawn from the point G to the line DLE, are equal. The line DLE is, therefore, the circumference of a circle having its centre G in the axis.

COR. The figure contained by the circle DLE, and that part of the conical surface which

which is intercepted between this circle and the point A, is a cone; and the common section of the cutting plane and the triangle passing through the axis, is a diameter of the circle DLE.

PROP. XXIV. [*Prop. 5. B. 1. Apoll.*]

If a scalene cone cut through the axis by a plane at right angles to the base, be cut also by another plane at right angles to the triangle passing through the axis; if this other plane cuts off, towards the vertex, a triangle similar to the triangle through the axis, but subcontrarily situated; the section made in the cone by this other plane is a circle.

Let there be a scalene cone, the vertex of which is the point A, and the base the circle BC; let it be cut through the axis by a plane perpendicular to the base, and let the section be the triangle ABC; let it be

Fig. 14.

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be also cut by another plane at right angles to the triangle ABC ; and let this other plane cut off, towards the vertex, the triangle AGK similar to the triangle ABC , but * subcontrarily situated; and let the section made in the surface be the line GKH : this line is the circumference of a circle.

In the lines GKH , BC take certain points H , L , from which let perpendiculars be drawn to the plane of the triangle ABC ; these perpendiculars will [38. 11. Elem.] fall on the common sections of the planes: accordingly, let them be HF , LM : HF , therefore, is parallel to [6. 11. Elem.] LM : next, through F draw DFE parallel to BC : the plane, therefore, which passes through FH , DE is parallel to the [15. 11. Elem.] base of the cone; and, for this reason, the section DHE is [23. 1.] a circle, of which DE is a diameter: the rectangle, therefore, contained by DF , FE is equal to the square of FH . And since ED is parallel to BC , the angle ADE is equal to the angle ABC ; and the angle AKG is placed equal to the angle ABC ; therefore the

* The meaning is this: The base GK is to be so placed, that it may make the angle AKG , and not the angle AGK , equal to the angle ABC .

angle AKG is also equal to ADE : and the angles at F are equal, for they are opposite vertical angles; therefore the triangle DFG is similar to the triangle KFE : therefore as EF to FK , so is GF to FD ; therefore the rectangle EFD is equal to the rectangle KFG . But the rectangle EFD , (that is, the rectangle contained by DF , FE ,) has been proved to be equal to the square of FH ; therefore the rectangle contained by KF , FG is equal to the same square of FH . It may, in like manner, be demonstrated, that the square of any straight line whatever, drawn from the line GK , so as to be perpendicular to GK , is equal to the rectangle contained by the segments into which that straight line divides the same GK : the section GK is, therefore, a circle having GK for a diameter *. — A section of this kind may be named a *subcontrary section*.

PROP. XXV.

If a cone cut through the axis by a plane, be cut likewise by ano-

* See the *lemma* placed at the end of this book.

ther plane, cutting its base in the direction of a straight line perpendicular to the base of the triangle passing through the axis; and if the common section of the triangle through the axis, and of the plane cutting the base of the cone in the direction of the perpendicular, be parallel to one of the sides of the triangle through the axis; the line which is the common section of the base-cutting plane, and the conical surface, is a parabola having for a diameter the straight line which is the common section of the triangle through the axis, and the same base-cutting plane.

Fig. 15. Let there be a cone, the vertex of which is the point A, and the base the circle BC; let it be cut through the axis by a plane, and let the section be the triangle ABC;

let

let it be also cut by another plane, cutting its base in the direction of the straight line DE perpendicular to the straight line BC; let the line DFE be the section made in its surface; and let FG, the common section of the triangle through the axis, and that other plane, be parallel to AC, one of the sides of that triangle: the line DFE is a parabola, and FG one of its diameters.

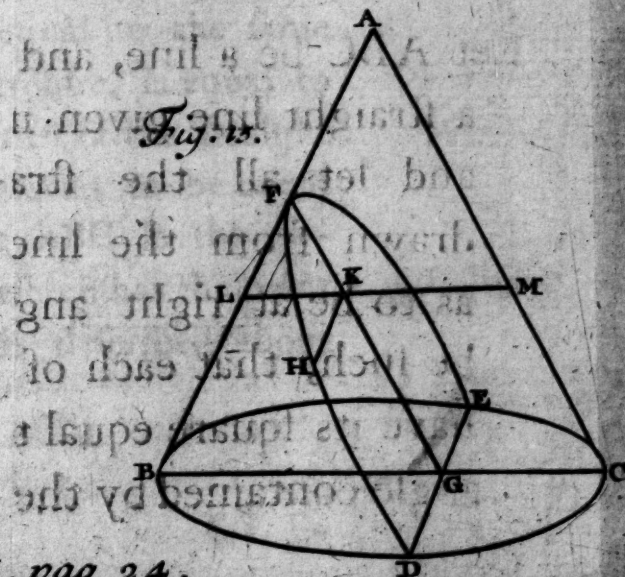
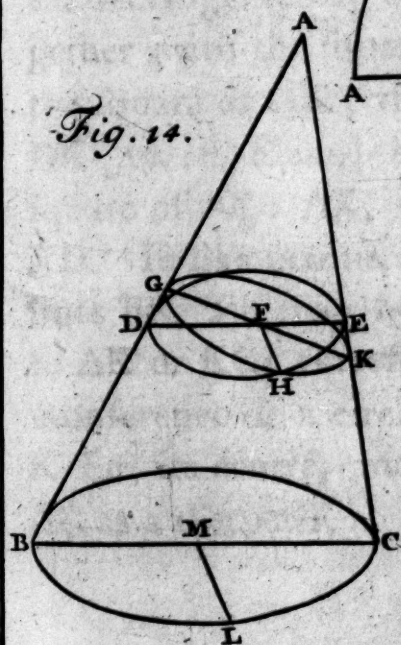
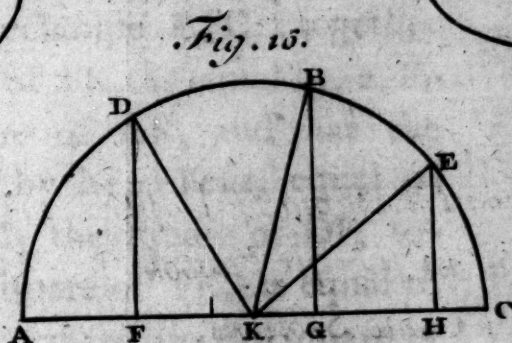
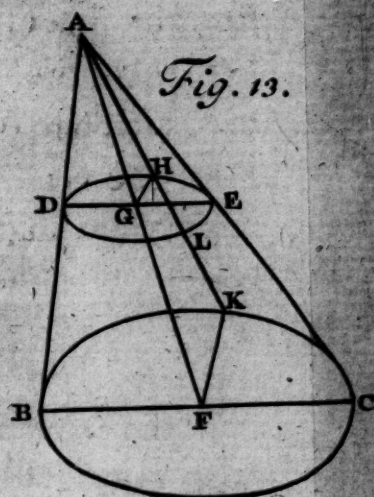
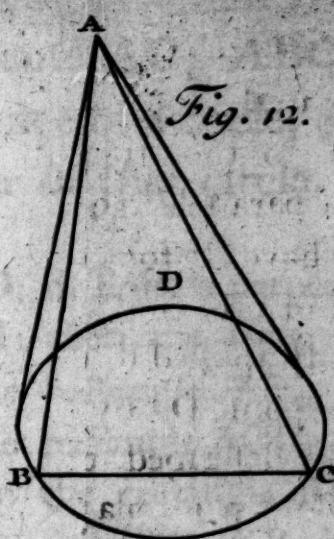
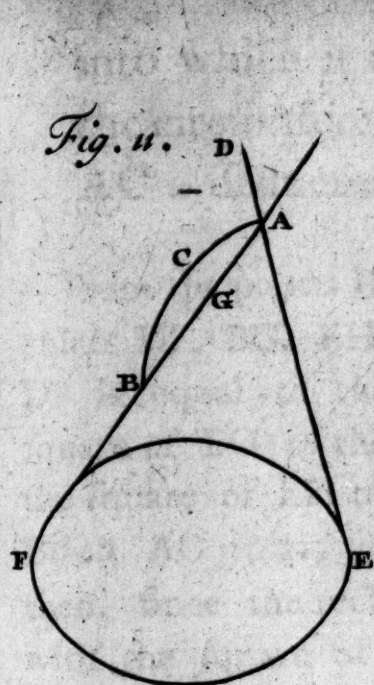
In the section DFE take any point H, and through H draw HK parallel to DE; produce HK to FG; and through K draw LM parallel to BC: therefore the plane passing through HK, LM is [15. 11. Elem.] parallel to the plane through DE, BC, that is, to the base of the cone: and, consequently, the plane through HK, LM is a [23. 1.] circle of which LM is a diameter. But HK is perpendicular to LM, [10. 11. Elem.], because DE is perpendicular to BC: therefore the rectangle LKM is equal to the square of HK [35. 3. Elem.]; and, in like manner, the rectangle BGC is equal to the square of DG: therefore the square of DG is to the square of HK, as the rectangle BGC to the rectangle LKM; and GC is equal to KM; therefore the rectangle BGC is to the rectangle LKM, as BG to

H 2
LK,

LK, that is, as GF to KF; therefore the square of DG is to the square of HK as the straight line GF to the straight line KF. Let, therefore, a parabola [19.1.] be described, which may have GF for a diameter, and F for the vertex of GF, and in which DG may be ordinately applied to the same GF: and because the point D, by construction, is in the parabola described, the point H is likewise in this same parabola [2 cor. 13. 1.]. And the same thing may be demonstrated with regard to all the points of the section DFE,

The second Lemma of Pappus, as it is extant in the first book of Apollonius's Conic Sections,

Fig. 16. Let ABC be a line, and let AC be a straight line given in position; and let all the straight lines drawn from the line ABC, so as to be at right angles to AC, be such, that each of them may have its square equal to the rectangle contained by the segments, into



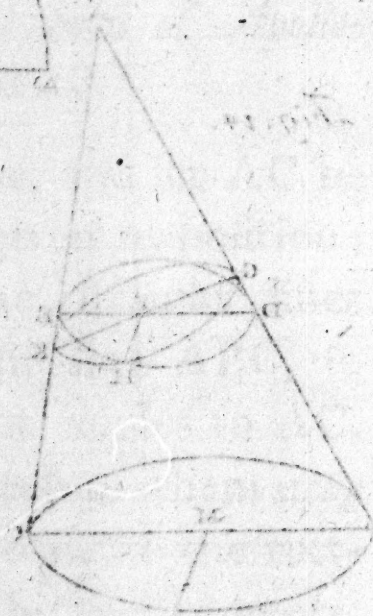
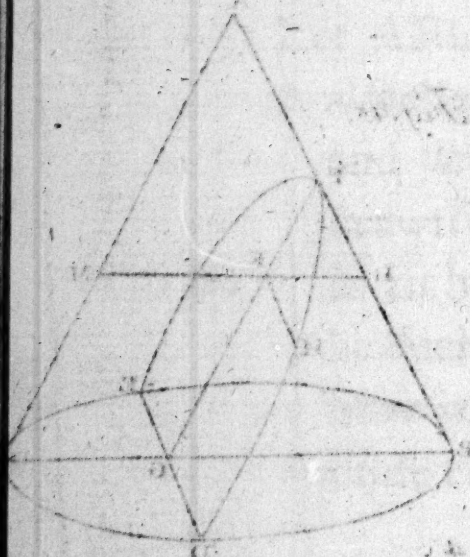
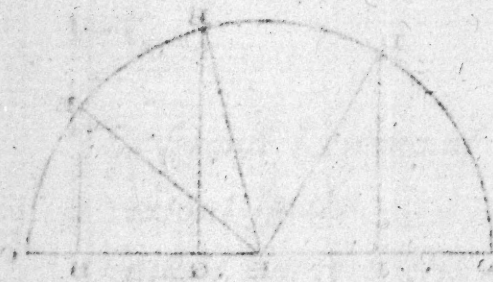
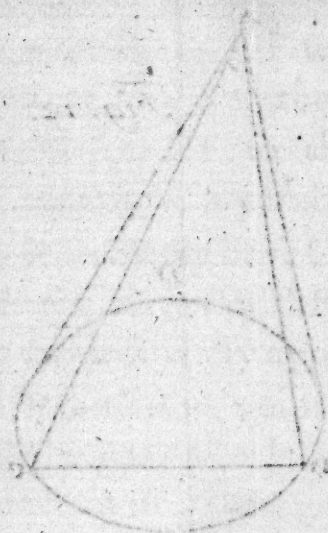
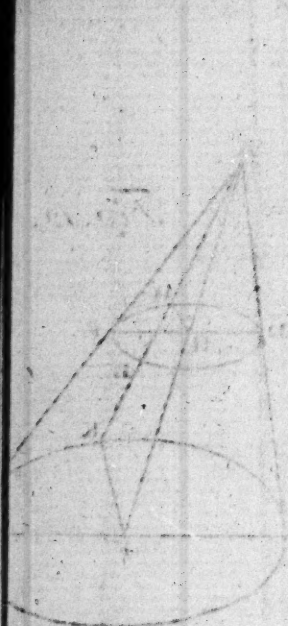


TABLE III

into which it cuts AC ; ABC is the circumference of a circle, and AC a diameter of that circle.

From the points D, B, E draw perpendiculars DF, BG, EH ; then the square of DF is equal to the rectangle AFC , the square of BG to the rectangle AGC , and the square of EH to the rectangle AHC . Bisect AC in K , joining KD, KB, KE : then, since the rectangle AFC , together with the square of FK , is equal [5. 2. Elem.] to the square of AK , and that the square of DF is, by hypothesis, equal to the rectangle AFC ; the square of DF , together with the square of FK , is equal to the square of AK ; therefore the square of DK [47. 1. Elem.] is equal to the same square of AK : AK , therefore, is equal to KD . In like manner, each of the straight lines BK, EK may be proved to be equal to AK or KC ; therefore ABC is the circumference of a circle which has the point K for its centre, and is described about AC as a diameter.

E L E M E N T S
OF THE
C O N I C S E C T I O N S.

B O O K II.
Of the ELLIPSE.

D E F I N I T I O N S.

I. **I**N two points D, E, taken in a plane, are fixed the ends of a string: the length of the string is greater than the distance between these points. A pin H, applied to the string, and held so as to keep it uniformly tense, is moved round, till it return to the place from which the motion has commenced; The point of the pin, as it moves round, describes upon the plane a line named the ELLIPSE.

II. The points D, E are named the *foci*.

III.

III. The point C which bisects the straight line between the foci, is named the *centre* of the ellipse.

IV. A straight line passing through the centre, and terminated both ways by the ellipse, is named a *diameter*; and the points where a diameter meets the ellipse, are named the *vertices* of that diameter.

V. The diameter which passes through the foci, is named the *greater axis*.

VI. The diameter perpendicular to the greater axis, is named the *less axis*.

VII. Two diameters, each of which bisects all straight lines in the ellipse that are parallel to the other, are named *conjugate diameters*.

VIII. A straight line not passing through the centre, but terminated both ways by the ellipse, and bisected by a diameter, is said to be *ordinately applied* to that diameter, or it is named, simply, an *ordinate* to that diameter. Also a diameter parallel to a straight line ordinately applied to another diameter, is said to be *ordinately applied* to that other diameter.

IX. A third proportional to two conjugate diameters is named the *latus rectum*,
or

of the *parameter*, of that diameter, which is the first of the three proportionals.

X. A straight line which meets the ellipse only in one point, is said to touch it in that point.

PROP. I. THEOR.

If two straight lines be drawn from any point in an ellipse to the foci, they are together equal to the greater axis.

Fig. 1.

HD, HE drawn from H, a point in an ellipse, to the foci D, E, are together equal to AB the greater axis.

Because H is a point in the ellipse, HD, HE are together equal to the length of the string with which it is described; and because the point A is likewise in the ellipse, DA, EA are together equal to the length of the string. For a like reason, EB, DB are together equal to the same length: DA, EA are, therefore, equal to EB, DB. Take away the common part DE, and the remainder, to wit, twice AD, will be equal to the remainder twice EB: AD, therefore, is equal to EB. Add the com-

mon part AE; and AD, together with AE, will be equal to the greater axis: but AD, together with AE, is equal to the length of the string, that is, to HD together with HE; therefore HD and HE are together equal to the greater axis AB.

COR. 1. The greater axis is bisected in the centre C. For since [def. 3.] DC is equal to EC, and that DA is equal to EB; AC is equal to CB.

COR. 2. Two straight lines drawn from a point without an ellipse to the foci, are together greater than the greater axis; but if drawn from a point within an ellipse to the foci, they are [20. 1. Elem.] together less than that axis.

COR. 3. A point is either in, without, or within, an ellipse, according as two straight lines drawn from it are either equal to, greater, or less, than the greater axis.

COR. 4. The distance of either vertex G of the less axis from either of the foci, is equal to half the greater axis. Join GD, GE: then, because CD is equal to CE, CG common, and the angles at C right angles; the triangle CDG is equal to the
I triangle

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triangle CEG; therefore DG is equal to EG: but DG and EG are together equal to the greater axis; therefore each of them is equal to the half of it.

COR. 5. The less axis FG is bisected in the centre. Draw straight lines DF, DG from the focus D to the vertices of the less axis; DF, DG, by the preceding corollary, will be equal; the angle DFC is, therefore, equal to DGC, and DCF, DCG are right angles; therefore CF is [26. 1. Elem.] equal to CG,

PROP. II. THEOR.

The square of half the less axis is equal to the rectangle contained by the segments of the greater axis, intercepted between the vertices of the greater axis and either of the foci.

Fig. 1. From either focus E to either vertex G of the less axis, draw the straight line GE; let C be the centre of the ellipse, and A, B the vertices of the greater axis: then the squares of GC, CE are together equal to the square of GE, that is, to the square
[4 cor.

[4 cor. 1. 2.] of CB, that is, to the rectangle AEB together with the square of EC [5. 2. Elem.]: take away the common square of CE, and the remaining square of GC will be equal to the remaining rectangle AEB.

PROP. III. THEOR.

Every diameter of an ellipse is bisected in the centre.

Let HK be a diameter; it is bisected Fig. 21
in C: for if CK be not equal to CH, let Ck be equal to CH; and from the points H, K, k draw straight lines to the foci D, E: then, because CD is equal to CE, and that Ck is placed equal to CH; the triangle DCH is equal to the triangle ECk, and the base DH to the base Ek. EH, in the same manner, is equal to Dk: therefore EK, DK are together equal to DH and HE together, that is, to EK and DK together: which is [21. 1. Elem.] absurd; therefore CH is equal to CK.

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PROP. IV. THEOR.

If a straight line be drawn from a point in an ellipse, so as to be at right angles to the greater axis; if another straight line be drawn from the same point to the nearer (to this point) of the foci; half the greater axis is to the distance of this focus from the centre, as the distance between the centre and the perpendicular is to the excess of half the greater axis above the straight line drawn to this same focus.

Fig. 1. From H, a point in an ellipse, let HK be drawn perpendicular to the greater axis AB, HE being drawn from the same point to the focus E: CB, the half of the greater axis, is to CE, the distance of the focus E from the centre, as CK, the distance between the centre C, and the perpendicular HK, is to the excess of CB above HE.

Having made BL equal to EH, and drawn HD to the other focus, describe from the centre H, distance HE, a circle meeting the axis AB again in O, and the straight line DH in the points M, N: then, because DE is the double of CE, and OE the double of [3. 3. Elem.] KE, the whole, or the remainder DO, is double the whole, or the remainder CK: and because DN is equal to DH together with HE, that is, to [1. 2.] AB, DN is the double of CB: but MN is the double of HE or LB; therefore the remainder DM is double the remainder CL: and because of the circle, the rectangle NDM is [cor. 36. 3. Elem.] equal to the rectangle EDO; therefore, as [16. 6. Elem.] ND or AB to DE, so is DO to DM: but the halves of magnitudes have the same ratio to one another which the wholes have: as therefore CB to CE, so is CK to CL, the excess of CB above LB, or HE.

COR. If in a straight line AB given in position and magnitude, and bisected in C, a point E be taken, between the points B, C; and from the point H, KH be drawn perpendicular to AB; and HE be joined,

Fig. 1.
n. 1.

joined, and BL placed equal to it towards the point C; and CL, CK, CE, BC be proportionals, and L, K be on the same side of C; the point H is in an ellipse given in position, that is, in an ellipse that has AB for the greater axis, and the point E for one of the foci.

Make AD equal to BE, and, as directed in the first definition, describe an ellipse that shall have AB for the greater axis, and the points D, E for the foci; the point H will be in that ellipse: for, if not, let KH meet the ellipse in the point Q; join EQ, and make BR equal to it: therefore, by the proposition, as CR to CK, so is CE to CB: but, by hypothesis, CL is to CK as CE to CB; therefore CR is to CK as CL to CK: CL, therefore, is equal to CR; which is absurd: the ellipse, therefore, meets not the straight line KH in Q; nor, as may in like manner be proved, does it meet HK any where on the same side of AB but in the point H. It therefore passes through the point H.

PROF.

PROP. V. THEOR.

The same construction remaining, if there be placed from the nearer (to the point taken in the ellipse) of the vertices of the greater axis, and towards the centre, a straight line, equal to the distance between the point in the ellipse and the nearer of the foci; the square of the perpendicular is equal to the excess of the rectangle contained by the segments into which the same perpendicular divides the greater axis, above the rectangle contained by the segments intercepted between the point where the placed line terminates towards the centre, and the foci.

From B, the nearer (to the point H) of the vertices of the greater axis, and towards the centre C, let BL be placed equal

Fig. 1.
D. 1. 2.

qual to HE; the square of the perpendicular HK is equal to the excess of the rectangle AKB, contained by the segments into which HK divides the greater axis, above the rectangle DLE, contained by the segments intercepted between L, where BL terminates, and the foci D, E.

For since the straight line CB is cut into any two parts in the point L, the squares of BC, CL are together equal [7. 2. Elem.] to twice the rectangle BCL, together with the square of BL, that is, to twice [preced. prop. and 16. 6. Elem.] the rectangle ECK, together with the square of BL, or HE, that is, to twice the rectangle ECK, together with the squares of KE, KH, that is, to the [7. 2. Elem.] squares of EC, CK, and KH together: the squares, therefore, of BC, CL are together equal to the squares of EC, CK, and KH together: but the squares of BC, CL are together equal [5. 2. and 2. ax. Elem.] to the rectangle AKB, together with the squares of CK, CL; and the squares of EC, CK, and KH are together equal [5. 2. and 2. ax. Elem.] to the rectangle DLE, together with the squares of CL, CK, KH; therefore the rectangle AKB, together with the squares of CK, CL,

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be parallel to the less axis FG , and meet AB the greater in K ; the square of AB is to the square of FG as the rectangle AKB to the square of HK . Having drawn to the foci D, E two straight lines HD, HE , place, from the vertex of the greater axis, and towards the centre C , the straight line BL equal to HE the lesser of them; and because CB, CE, CK, CL are [4. 2.] proportionals, their squares are also proportionals: but the square of CB is [5. 2. Elem.] made up of the square of CK and the rectangle AKB ; and the square of CE of the [5. 2. Elem.] square of CL and the rectangle DLE ; therefore the square of CB is [19. 5. Elem.] to the square of CE , as the remaining rectangle AKB to the remaining rectangle DLE ; and, by conversion, the square of CB is to the rectangle AEB , as the rectangle AKB to its excess above the rectangle DLE , that is, as the rectangle AKB to [preceding prop.] the square of KH : but [2. 2.] the rectangle AEB is equal to the square of CF : as, therefore, the square of CB to the square of CF , so is the rectangle AKB to the square of KH ; the square, therefore, of AB is to that of FG ,

FG, as the rectangle AKB to the square of HK.

In the other case, let HP be the straight line drawn from the point parallel to the greater axis, and meeting the lesser in P; the square of FG is to the square of AB, &c. The square of CB, as hath been proved, is to the square of CF, as the rectangle AKB to the square of HK or PC: therefore the square of CF [prop. B. and 19. 5. Elem.] is to the square of CB, as the rectangle FPG to the square of CK, that is, as the rectangle FPG to the square of HP.

Fig. 1.
n. 1. 2.

COR. 1. Hence the squares of straight lines drawn from points of an ellipse to either axis, so as to be parallel to the other axis, are to one another as the rectangles contained by the segments of the axis met, which are between the straight lines and its vertices. For let HM, PQ be parallel to the axis FG, and meet the other axis in M, Q; and, by the proposition, the rectangle AMB is to the square of HM as the square of AB to the square of FG, that is, as the rectangle AQB to the square of PQ; and, alternately, the rectangle

Fig. 2.

K 2

AMB

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AMB is to the rectangle AQB, as the square of HM to the square of PQ.

Fig. 2. COR. 2. If a circle be described upon either axis as a diameter, and MH, QP parallel to the other axis, meet it in the points N, R; the segments (of these parallels) between the axis, and the circumference of the circle, are to one another as the segments between the axis and the ellipse: for the rectangles AMB, AQB are equal [35. 3. Elem.] to the squares of MN, QR, each to each: therefore the square of MN is to the square of QR, as the square of MH to the square of QP; consequently the straight lines MN, QR, MH, QP are also [22. 6. Elem.] proportionals.

PROP. VII. THEOR.

Every straight line terminated both ways in an ellipse, and parallel to either axis, is bisected by the other axis; or, what is the same thing, the axes are conjugate diameters.

Let

Let HL be parallel to the axis FG, and meet the other axis AB in M, and the circle described upon AB in the points N, O; then, as MN to MO, [2 cor. preceding prop.], so is MH to ML; and because NO is cut at right angles by AB, MN is equal to [3. 3. Elem.] MO; therefore MH is equal to ML.

Fig. 2.

PROP. VIII. THEOR.

Every straight line terminated both ways in an ellipse, and bisected by one axis, is parallel to the other.

HP bisected in S by the axis FG, is parallel to the other axis AB. Draw HM, PQ parallel to FC, and let them meet the circle described upon AB in the points N, R; then, because HM, FC, PQ are parallels, as HS to SP, so is MC to CQ: but HS is equal to SP; therefore, also, MC is equal to CQ; consequently MN is [14. 3. Elem.] equal to QR; and therefore HM is equal to [2 cor. 6. 2.] PQ: but HM is also parallel to PQ; therefore HP is [3. 3. 1. Elem.] parallel to MQ.

Fig. 2.

COR.

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COR. Hence straight lines HM , PQ , parallel to either axis FG , and which cut off between the centre and the points where they terminate in the other axis, equal segments MC , QC of this other axis, are equal to each other: and, on the contrary, if HM , PQ be equal to each other, and parallel to either axis, they cut off [2 cor. 6. 2.—14. 3. Elem.] equal segments MC , QC of the other axis.

PROP. IX. THEOR.

Of all diameters the greater axis is the greatest, and the less axis the least; and a diameter which is nearer to the greater axis, is greater than one more remote.

Fig. 2.

Let CB be half the greater axis, and FC half the less; let CH be any other semi-diameter, and draw HM at right angles to CB : and because the square of CB is to the square of CF as the rectangle AMB to the square of HM , and that [4. cor. 1. 2.] CB is greater than CF ; therefore the rectangle AMB is greater than the square of HM :

HM: to these unequals add the common square of CM; and the square of CB [5. 2. Elem.] will be greater than that of CH [47. 1. Elem.]; and consequently the straight line CB will be greater than CH. Next draw HS at right angles to the less axis, and it may, in the same manner, be demonstrated, that CF is less than CH: therefore, of all semidiameters, CB is the greatest, and CF the least.

Let now CT be more remote from the greater axis than CH; CH is greater than CT: draw TV parallel to HM, and let it meet AB in V, and the ellipse again in H, and let HZ be parallel to MV, and make CQ equal to CM. Fig. 2.

Because the rectangle AVB, which consists of the * rectangles AMB and QVM, is

* This is demonstrated in prop. 31. book 7. of Pappus of Alexandria. The proposition is to this purpose. "If in a straight line AB two equal straight lines AQ, BM be taken, and betwixt Q and M a point V; the rectangle AVB is equal to the rectangles AMB, QVM together." For AB being bisected in C, the rectangle AVB, together with the square of CV, is [5. 2. Elem.] equal to the square of CB, that is, to the rectangle AMB, together with the square of CM [5. 2. Elem.] that is, to the rectangle AMB, together with the

See fig. 2.
of this
book.

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is to the square of VT , which [5. 2. Elem.] is made up of the square VZ and rectangle TZX , as [1. cor. 6. 2.] the rectangle AMB to the square of MH or VZ ; the whole the rectangle AVB , is to the whole the square of VT , as [19. 5. Elem.] the remaining rectangle QVM to the remaining rectangle TZX : but the rectangle AVB is [35. 3. Elem.] greater than the square of VT ; therefore the rectangle QVM is also greater than TZX . To these unequals add the square of CV , and the square of CM [5. 2. Elem.] will be greater than the rectangle TZX together with the square of CV . Superadd to the same unequals, the square of MH or ZV , and the squares of CM , MH together, will be greater than the squares of VT , VC together; that is, the square of CH is greater than the square of CT ; and therefore CH is greater than CT .

the rectangle QVM , and the square of CV [5. 2. Elem.]. Take away the common square of CV , and the remaining rectangle AVB is equal to both the remaining rectangles AMB , QVM .

L E M.

LEMMA I.

If a point A be taken in a straight line AE, and two parallels BC, DE be drawn on the same side of AE, and on the same side of the point A, or on the contrary sides of AE, and on the contrary sides of the point A; if these parallels have the same ratio to each other, as the segments of AE, which are intercepted between them and the point A; the extremities B, D of the parallels and the point A are in the same straight line.

Fig. 3. 4.

In the one case, complete the parallelograms AB, AD; these are similar, and have the angle at A common: consequently they are about the same [26. 6. Elem.] diameter, that is, the points A, B, D are in the same straight line.

Fig. 3.

In the other case, join BA, DA: and because BC is to CA as DE to EA, and the

Fig. 4.

L

angle

angle BCA equal to DEA, the triangles ABC, ADE are equiangular, [6. 6. Elem.]; consequently the angle EAD is equal to the angle CAB: and therefore AB, AD are [14. 1. Elem.] in the same straight line.

LEMMA II.

Fig. 5. 6.
7. 8.

If two parallel straight lines AC, BD be drawn from two points A, B, and other two parallels AE, BF, either coinciding or not coinciding with the parallels AC, BD, and having the same ratio to each other as the parallels AC, BD, and lying on the same side of AB, or on the contrary sides of it, according as the same parallels AC, BD are on the same side of AB or on the contrary sides of it, be placed from the same points; if a straight line be drawn through the extremities of either of the pairs

pairs of parallels, the point where it meets AB, and the extremities of the other two parallels, are in the same straight line.

Let a straight line be drawn, for example, through C, D the extremities of AC, BD, and let it meet the straight line AB in the point G; the extremities E, F of the other two parallels AE, BF, and the point G, are in the same straight line. For since the triangles AGC, BGD are equiangular, AG is to BG as AC to BD: but AC is to BD (by hypothesis) as AE to BF; therefore AG is to BG as AE to BF: consequently, by the preceding lemma, the points G, E, F are in one and the same straight line. In like manner, if a straight line be drawn through the extremities E, F of the parallels AE, BF, the point where it meets AB, and the extremities C, D of the other two parallels, are in the same straight line.

L 2 . I. E M-

L E M M A III.

The same things being still supposed which are supposed in the second lemma, if through the extremities of either of the pairs of parallels two straight lines be drawn parallel to each other, and meeting AB; the straight lines that join the two points where AB is met, and the corresponding extremities of the other two parallels, are likewise parallel.

Fig. 9
10. 11.

For example, let CH, DK be drawn parallel to each other through the extremities CD of the parallels AC, BD; and let them meet A, B in the points H, K; the straight lines EH, FK, which join the extremities of the other two parallels AE, BF, and the corresponding points H, K in the straight line AB, are likewise parallel.

Because both AC, BD, and CH, DK, are parallels, the triangles ACH, BDK are equiangular; therefore AH is to BK

as

Fig. 1. n. 1.

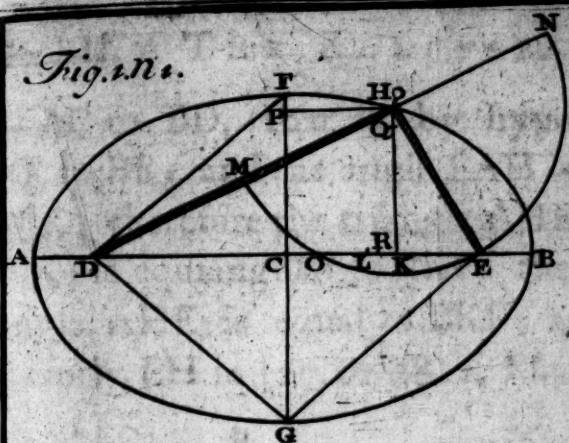


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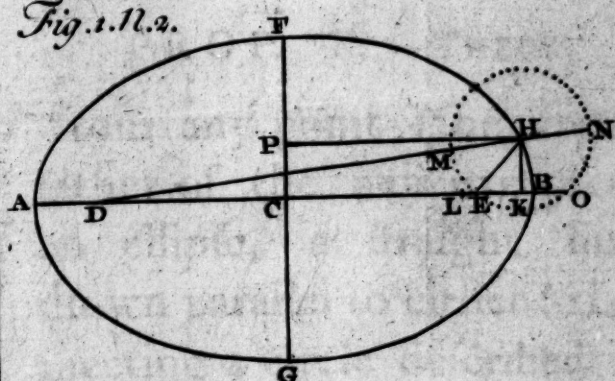


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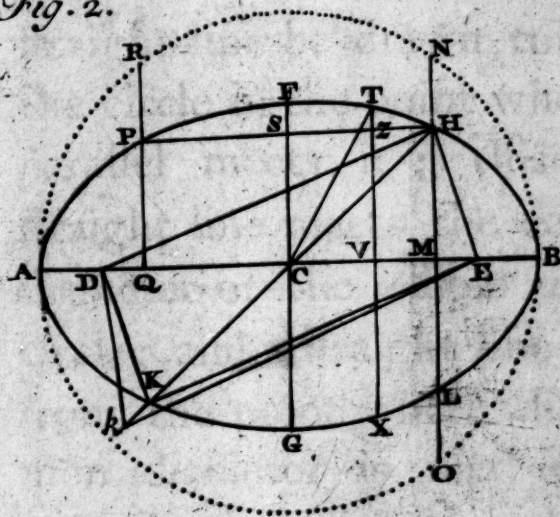


Fig. 3.

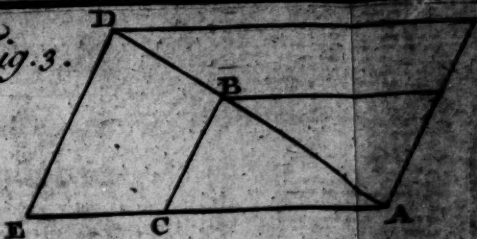


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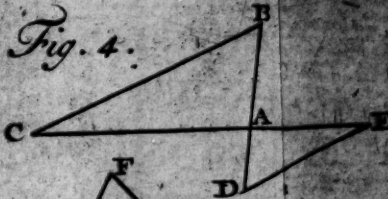


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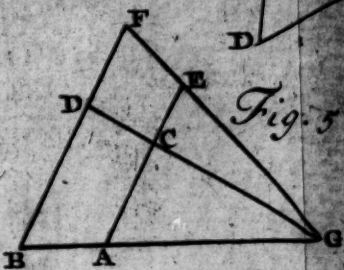


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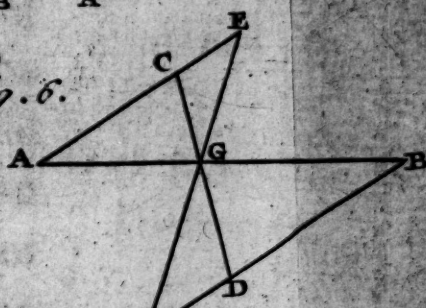


Fig. 7.

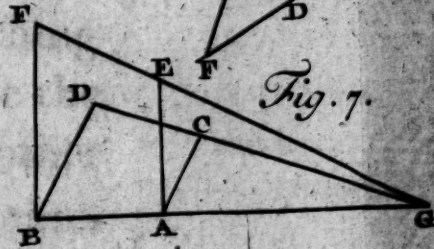
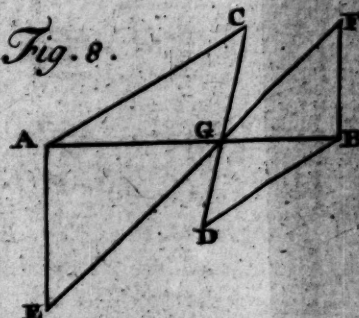


Fig. 8.



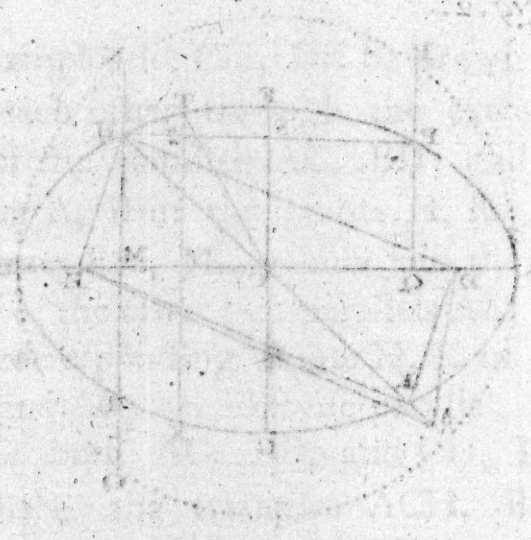
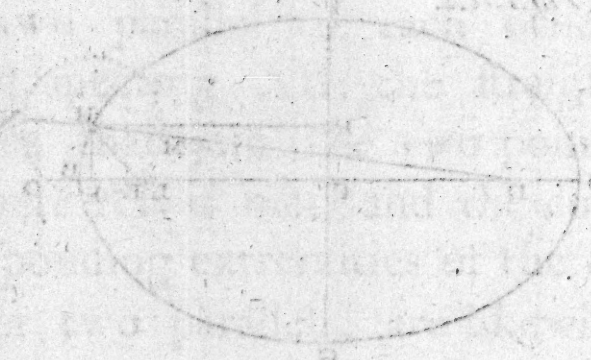
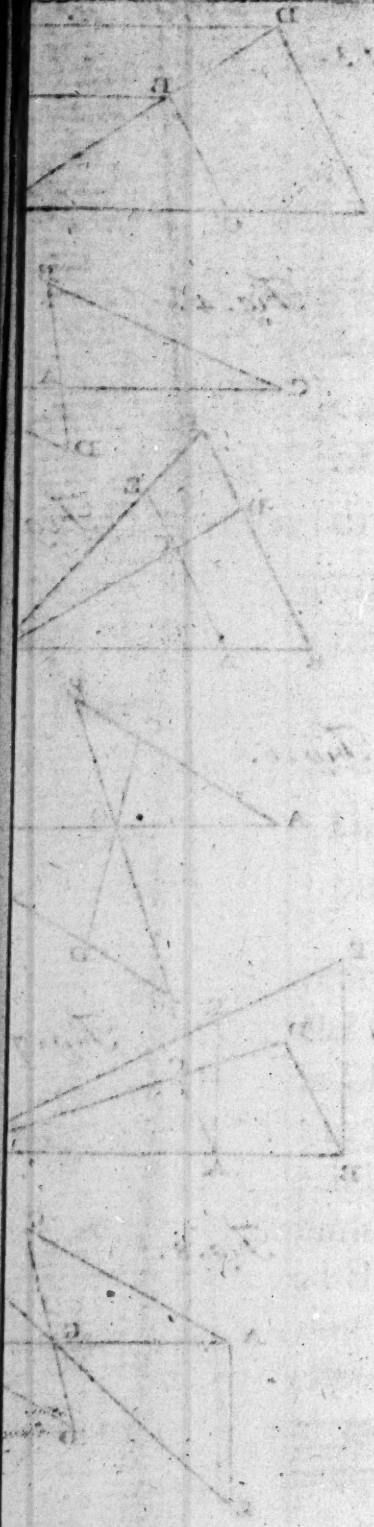


TABLE III



as AC to BD, that is, by hypothesis, as AE to BF; and the angle EAH is equal to FBK: therefore the triangle AHE [6. 6. Elem.] is equiangular to BKF: therefore the angle AHE is equal to BKF; and consequently EH is [27. or 28. 1. Elem.] parallel to FK.

PROP. X. THEOR.

If from any point (the vertex of either of the axis excepted) of an ellipse, a straight line be drawn parallel to either axis, and meeting a circle described upon the other axis, and another straight line be drawn touching the circle in the point where the parallel meets it; this other straight line meets the common diameter of the ellipse and the circle; and a straight line drawn from the point where the common diameter is met, to the point in the ellipse, touches the ellipse:

ellipse: also a straight line which is drawn through the vertex of either axis parallel to the other axis, touches the ellipse.

Fig. 12. Let E be the point in the ellipse, and EF a straight line drawn parallel to either axis CD , and meeting a circle described upon AB in G ; let GN touch the circle in G , and meet the common diameter AB in H ; the straight line that joins the point H and the point taken in the ellipse touches the ellipse.

Let C be the centre of the ellipse and circle, and join CG ; then because CGN is a right angle, and GCB less than the right angle DCB , GN , CB will necessarily meet: let them meet in H , and join HE ; HE will touch the ellipse: for if not, let it, if possible, meet the ellipse again in K , and through K draw a straight line parallel to EF , and let the parallel through K meet AB in L , and the circle in M : then, because EF is to KL as GF to ML , [2. cor. 6. 2.] and that the points H , K , E are in one straight line; the points H , M , G are likewise in one [lemma 2.] straight line: the
straight

straight line, therefore, HG meets the circle in two points G and M : but, by hypothesis, the same HG touches the circle; which is absurd: HE , therefore, meets not the ellipse any where but in the point E ; and therefore it touches it in this point.

Next, through D , the vertex of either axis CD , let a straight line be drawn parallel to the other axis AB ; this straight line touches the ellipse: for if not, let it, if possible, meet the ellipse again in O ; and through O draw a straight line parallel to CD ; and let the parallel through O meet AB in P , and the circle in Q ; also let CD meet the circle in R : then, because CO is a parallelogram, CD is equal to PO : but as CD to PO , so is CR to PQ , [2. cor. 6. 2.]; CR , therefore, and PQ are equal: but they are also parallel; therefore, if QR be joined, it will be parallel to CP ; and the angle QRC is, therefore, a right angle. And thus RQ , drawn perpendicular to a diameter of the circle from the extremity of that diameter, falls within the circle; which is absurd, [16. 3. Elem.]: therefore DO touches the ellipse.

Fig. 12.

COR.

COR. 1. If EH touch the ellipse, and GH be joined, it may be shewn in the same manner that GH touches the circle.

COR. 2. From a point in an ellipse, only one straight line can be drawn which will touch that ellipse: were it possible for two straight lines to touch the ellipse in one and the same point, two could also touch the circle in the same point; and thus the cor. to 16. 3. Elem. would be false.

COR. 3. A straight line cannot meet an ellipse in more than two points; for could a straight line meet the ellipse in more than two points, a straight line could also meet the circle in more than two points; which would be an absurdity: The ellipse, therefore, is convex on the side where straight lines touch it, but concave on the contrary side.

COR. 4. An angle contained by any diameter, (the axis excepted), and a straight line that touches the ellipse in the vertex of that diameter, and is drawn towards the greater axis, is greater than a right angle. Let CE be the diameter, EH the tangent, and AB the greater axis, the other

ther things remaining as in the proposition; the angle CEH is greater than CGH , that is, it is [21. 1. Elem.] greater than a right angle.

COR. 5. The proposition points out a method by which, if the greater axis be given in position and magnitude, a straight line can be drawn that will touch an ellipse in a given point.

COR. 6. Two straight lines touching an ellipse in the vertices of a diameter are parallel.

Let EH , ST touch the ellipse in the vertices of the diameter ECS , and let them meet the axis AB in H , T ; draw EF , SV perpendicular to the same axis, and let EF , SV meet the circle described upon it in G , X ; join GH , XT , which, by cor. 1. will touch the circle: and because GF , XV are divided in the same ratio, [2. cor. 6. 2.] in the points E , S , and that ECS is a straight line; therefore the points G , C , X are also in a straight line [lem. 2.]: and because GH , XT , which touch the circle in the vertices of the diameter GX , are parallels, EH , ST are also parallels, [lem. 3.]

Fig. 12.

M

PROP.

PROP. XI. THEOR.

If from a point of an ellipse two straight lines be drawn to the foci, and a straight line be drawn bisecting the angle adjacent to that contained by those two straight lines; that straight line touches the ellipse.

Fig. 13.

Case 1. When the straight line bisecting the angle is parallel to the greater axis, let AB be the greater axis, C the centre, and D, E the foci; from a point F of the ellipse draw the straight lines FD, FE ; let FH , which bisects the angle EFG , be parallel to the greater axis AB ; FH touches the ellipse: making FG equal to FE , join EG , and let it meet FH in H ; then, since FE is equal to FG , and FH common, and the angle EFH equal to GFH , EH is equal to HG : and FH, ED being parallels, EH is to HG as DF to FG ; DF , therefore, is equal to FG , that is to FE ; and in the triangles DFC, EFC , FC is common, and DC equal to CE ; therefore the angles at C

C are equal: and thus each of them is a right angle; therefore CF is [def. 6. 2.] the half of the less axis; and, consequently, FH, which is parallel to the other axis, touches [10. 2.] the ellipse.

Case 2. The construction remaining the same in other respects as in the preceding case, let now FH meet the greater axis in K, and draw FL at right angles to the same axis, and let it meet the circle described upon AB in the point M: joining MK, and drawing MC to the centre, make BN equal to EF; AN will consequently [1. 2.] be equal to DF: and since the outward angle EFG of the triangle DFE is divided into two equal parts by the straight line FK, which meets the base DE in the point K, DK is to KE as DF to FE [A. 6. Elem.], that is, as AN to NB: and, by composition, DK together with KE is to KE as AB to NB: take the halves of the antecedents, and CK will be to KE as CB to BN; and, by conversion and alternation, CK is to CB as CE to CN, that is, as CB to CL [4. 2.]: but CB is equal to CM; therefore, as CK to CM, so is CM to CL: therefore the triangle CMK is equiangular [6. 6 Elem.] to CLM: but CLM is a right angle;

Fig. 14.

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therefore the angle CMK is also a right angle; and consequently the straight line MK touches the circle, [16. 3. Elem.]; therefore FK touches the ellipse.

Otherwise : produce DF to G , so that FE may be equal to FG ; and in FK take any point O , and join OD , OE , OG , and EG , and let this last meet FK in P ; then, because FE is equal to FG , FP common, and the angle EFP equal to GFP ; EP , GP are equal, and the angles at P right angles; therefore OE is equal to OG : but DO , OG are together greater than DG ; consequently DO , OE are together greater than the same DG , that is, than DF , FE together, that is, than the greater axis AB : the point O is, consequently, without the ellipse, [3. cor. 1. 2.]; and, consequently, FK touches the ellipse.

COR. On the contrary, if a straight line FK touch the ellipse, and DF , FE be drawn from the point of contact to the foci; the angles DFO , EFK , which DF , FE make with the touching line in contrary directions of it, are equal. For let them be unequal; then DF being produced to G , the angles EFK , GFK are likewise unequal:
but

but by the proposition, a straight line which bisects the angle EFG touches the ellipse; and by the hypothesis, FK touches it in the point F; which is [2. cor. 10. 2.] absurd.

PROP. XII. PROB.

The greater axis of an ellipse being given in position and magnitude, and the foci being given, to draw a straight line parallel to another straight line given in position, and which shall touch the ellipse.

Let AB be the greater axis, D, E the foci; let ST be given in position: from E, either of the foci, draw EV at right angles to ST, and from the centre D, with a distance equal to AB, describe a circle: this circle will meet EV in two points; of which let G be either; and having joined DG, draw EF to it, and let EF make the angle GEF equal to EGD; and through F draw FK parallel to ST; FK touches the ellipse in the point F.

Fig. 14.

Because

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Because the angle GEF is equal to EGF , EF is equal to FG ; therefore DF , FE are together equal to DG , that is, to the greater axis AB : the point F is, therefore, in [3. cor. 1. 2.] the ellipse. Let FK meet the straight line VE in P ; and because FK , ST are parallels, EV at right angles to ST is likewise at right angles to FP : therefore the angles EFP , GFP are equal; and therefore FK touches the ellipse. In like manner, by employing the other point, where the circle described from the centre D meets EV , another tangent can be drawn parallel to ST .

PROP. XIII. THEOR.

Every straight line parallel to a straight line that touches the ellipse, and terminated both ways by the ellipse, is bisected by the diameter that passes through the point of contact.

If the diameter be either of the axes, a tangent drawn through its vertex is parallel to the other axis, [10. and 2. cor. 10. 2.];
and

and thus a straight line parallel to the tangent is also parallel to this other axis; and consequently is bisected by the diameter (or axis) that passes through the point of contact [7. 2.].

But let CE be any other diameter, and in its vertex let EF touch the ellipse; let GH , parallel to EF , be terminated in G , H , two points of the ellipse; and let GH meet the diameter CE in K ; GK is equal to KH .

Fig. 15.
n. 1, 2.

Let the tangent EF meet either axis AB in F ; let GH meet the same axis in L ; through the points E , G , H draw EM , GN , HO perpendiculars to AB , and let them meet the circle described upon AB in the points P , Q , R ; join FP , LQ , CP ; let LQ meet CP in the point S , and join SK : then, because EF touches the ellipse, FP touches the circle; and because the parallels QN , RO are cut in the same ratio in the points G , H , and that HGL is a straight line, the points L , Q , R are in a straight line [lemma 2.]. Again, since the parallels PM , QN are cut in the same ratio in the points E , G , through which the parallels EF , GL are drawn, FP , LQ are likewise parallels [lem. 3.]; CS , consequently,

frequently, is to SP as CL to LF, that is, as CK to KE: therefore KS is [2. 6. Elem.] parallel to EP, and consequently to QN, RO; therefore GK is to KH as QS to SR: but QS is equal to SR, because CP being at right angles to PF, which touches the circle, is also at right angles to RQ, which is parallel to PF; therefore GK is equal to KH.

COR. 1. On the contrary, any straight line GH, terminated both ways by the ellipse, and bisected by the diameter CE; or, in other words, any straight line ordinately applied to the diameter CE, is parallel to a straight line EF, which touches the ellipse in the vertex of this diameter. For, if not, draw a tangent that shall be [12. 2.] parallel to GH; and GH will be bisected by the diameter which passes through the point of contact: but, by hypothesis, the same GH is bisected by another diameter CE; which is absurd.

COR. 2. All straight lines ordinately applied to the same diameter are parallel to one another.

COR. 3. If two or more parallels be terminated both ways by an ellipse, the diameter

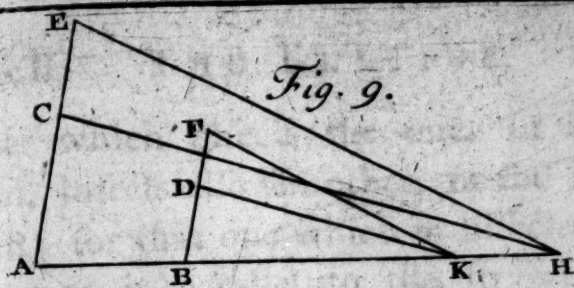


Fig. 13.

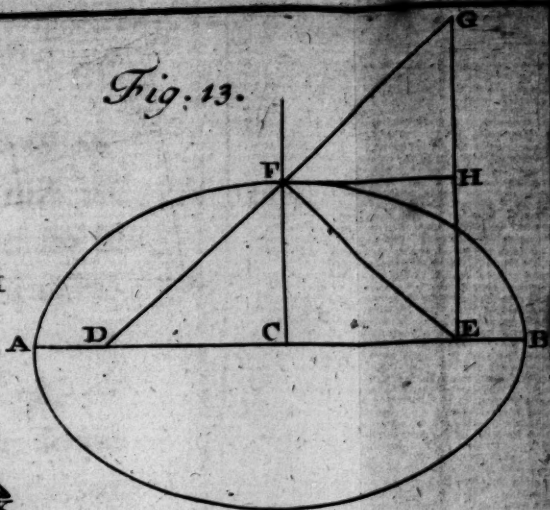


Fig. 10.

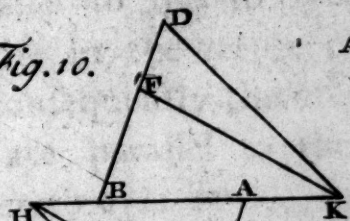


Fig. 11.

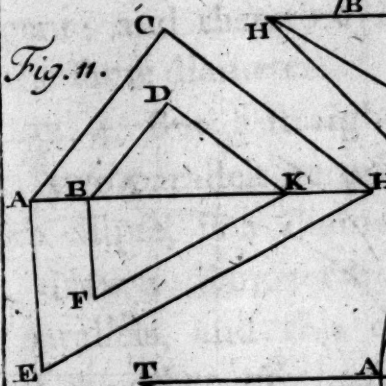


Fig. 12.

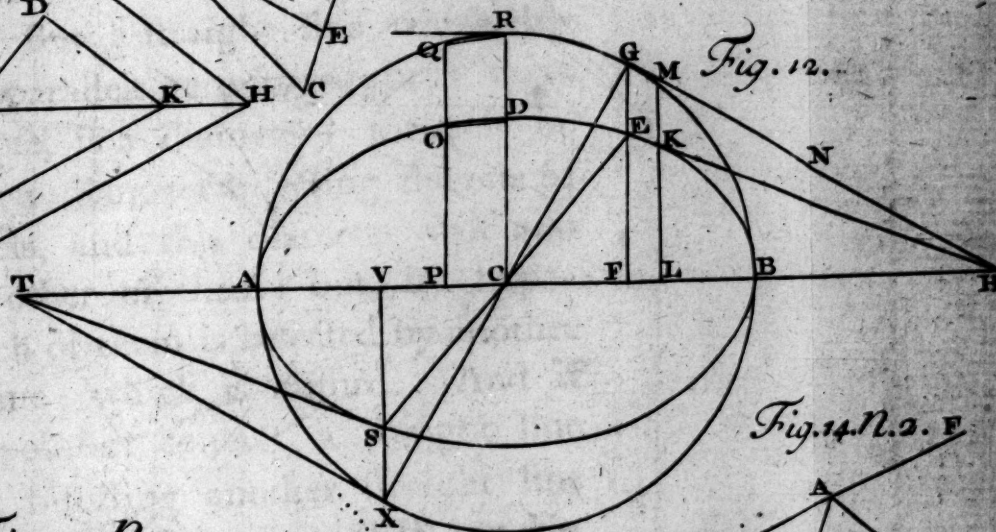


Fig. 14.N.1.

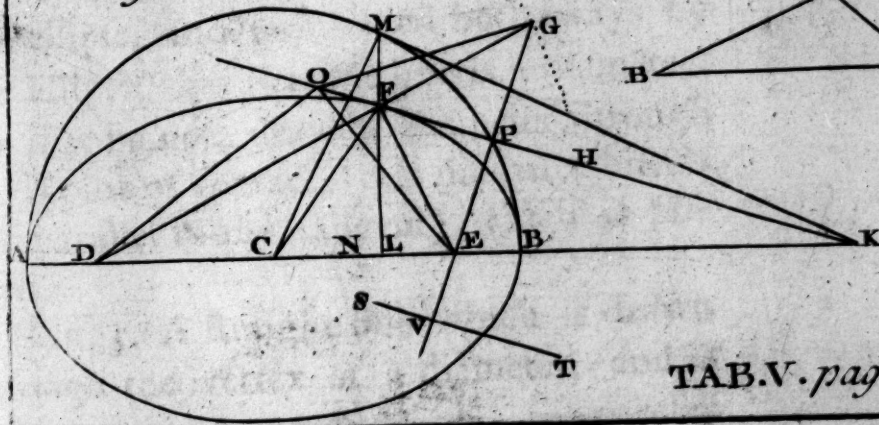
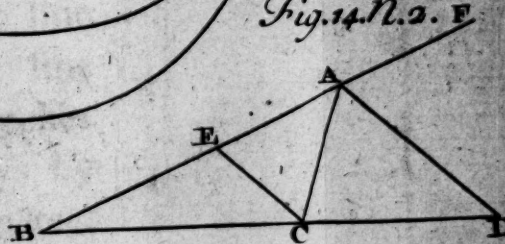
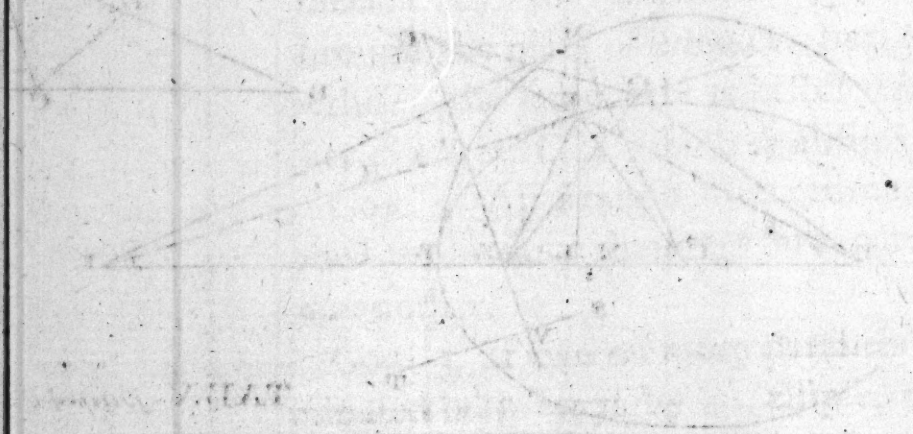
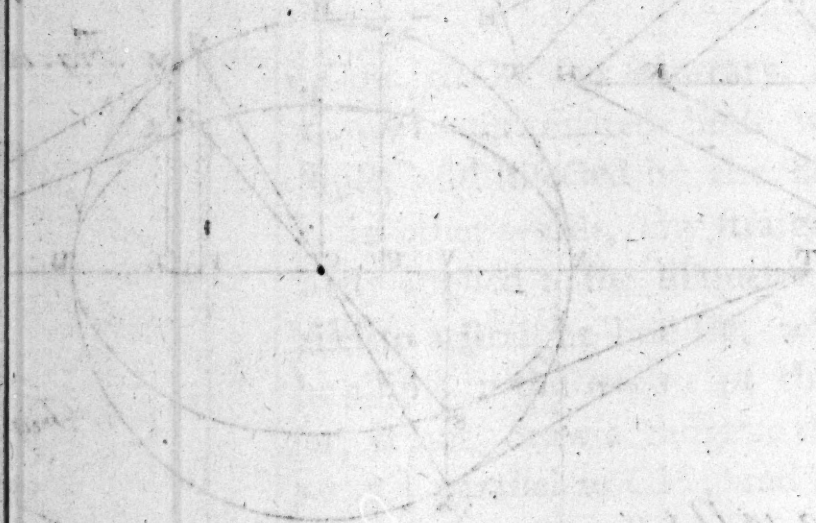
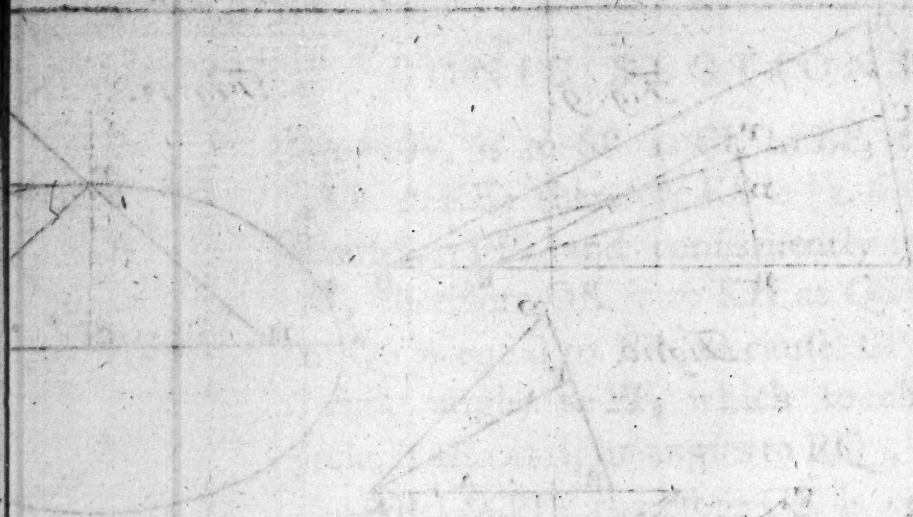


Fig. 14.N.2.





meter which bisects the one, or one of them, bisects also the other, or the rest of them: for that one which is bisected by a diameter, is parallel to the straight line touching the ellipse in the vertex of that diameter; and consequently the other, or the rest, is or are parallel to the same tangent; and therefore is or are bisected by the same diameter.

COR. 4. But a straight line which bisects two parallels terminated both ways by an ellipse, is a diameter: for if it be not, draw a diameter bisecting the one of the parallels, and this diameter will also bisect the other of them: but, by hypothesis, each of them is bisected by another straight line; which is absurd. And if from the point of contact, a straight line be drawn bisecting another straight line parallel to a straight line that touches the ellipse, and terminated both ways by the ellipse, that straight line is a diameter: for if it be not, draw a diameter through the point of contact; this diameter bisects the parallel to the tangent; which is absurd.

COR. 5. A straight line which is drawn through the vertex of a diameter, and is parallel

parallel to an ordinate to that diameter, touches the ellipse.

COR. 6. Two straight lines in an ellipse, which pass not through the centre, do not bisect each other: for if they bisected each other, they would be each of them parallel to a straight line touching the ellipse in the vertex of the diameter drawn through the point of bisection, and of consequence they would be parallel to each other; which is absurd.

PROP. XIV. THEOR.

Two diameters of an ellipse, either of which is parallel to a straight line touching the ellipse in either vertex of the other, are conjugate diameters.

Fig. 15.
n. 1. 2.

Let ET , VX be two diameters; let either of them, VX , be parallel to EF , a straight line touching the ellipse in the vertex E of the other; ET , VX are conjugate diameters.

Through the vertices E , V draw to either axis AB straight lines EM , VY parallel

lel to the other axis CD ; and let EM , VY meet the circle described upon AB in the points P , Z , on the same sides of the circle on which are the points E , V ; and from the point F , where EF meets the axis AB , draw the straight line FP , which will touch the [1. cor. 10. 2.] circle in P ; and from the point Z , the straight line Za touching the circle, and meeting the axis in a ; join aV ; aV will touch the [10. 2.] ellipse; and, last of all, from the centre draw CP , CZ .

Then, because the parallels PM , ZY are cut in the same ratio in the points E , V through which are drawn the parallels EF , VC ; FP , CZ are [lem. 3.] also parallels: but CPF is a right angle; consequently PCZ is also a right angle: and CZa is a right angle; therefore CP , Za are parallels: and consequently CE and aV are parallels: therefore straight lines parallel to CE will likewise be parallel to Va , which touches the ellipse in V ; and of consequence they will be bisected [13. 2.] by the diameter CV : and because CV , by hypothesis, is parallel to EF , all straight lines parallel to CV will likewise be parallel to EF , and will consequently be bisected by

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the diameter CE ; therefore CE , CV are conjugate diameters [7. def. 2.]

COR. 1. And since CV is the only diameter that can bisect straight lines parallel to CE , and terminated both ways by the ellipse, CV alone is conjugated to CE .

COR. 2. If, therefore, CE , CV be conjugate diameters, each of them is parallel to the tangent drawn through the vertex of the other.

COR. 3. On the other hand, a straight line which, when drawn through the vertex of a diameter, is parallel to the diameter conjugated to that diameter, touches the ellipse in that vertex.

COR. 4. A straight line parallel to a diameter, and terminated both ways by the ellipse, is bisected by the diameter conjugated to that diameter: for it is parallel to the straight line which touches the ellipse in the vertex of this diameter conjugated to the other; and consequently it is bisected by this same conjugated diameter. On the other hand, a straight line bisected by a diameter, is parallel to the diameter conjugated to that diameter.

P R O P.

PROP. XV. THEOR.

If from a point in an ellipse to either of two conjugate diameters, a straight line be drawn parallel to the other, the square of the diameter to which the straight line is drawn, is to the square of the other diameter, as the rectangle contained by the segments into which the straight line divides the diameter to which it is drawn, is to the square of that straight line.

Let ET , VX be conjugate diameters, and from a point G of the ellipse draw GK to ET , parallel to VX ; the square of ET is to the square of VX as the rectangle EKT to the square of GK .

Fig. 15.
n. 1. 2.

For since ET , VX are conjugate diameters, VX is parallel to the tangent EF , drawn through the vertex E of ET : and the same construction as in the two foregoing propositions still remaining, and what

what was there demonstrated being still kept in view, let SK, when produced, meet AB in the point β ; and through G draw the straight line $G\gamma$ parallel to the same AB, and meeting $S\beta$ in γ ; then, because PCZ is a right angle, and that the angles CPM, MCP are together equal to a right angle; the angle PCZ is equal to the same CPM and MCP together: take away the common angle MCP, and the remaining angle MCZ is equal to the remaining CPM; and CP, CZ are equal; therefore the right-angled triangles PMC, CYZ are equal; PM is, therefore, equal to CY; but since PM, $S\beta$, and PF, SL are parallels, the triangles PFM, $SL\beta$ are equiangular: and thus CPM, $SL\beta$ are [8. 6. Elem.] also equiangular; therefore CP is to PM as SL to $L\beta$, that is, as SQ to $N\beta$ [2. 6. Elem.]; alternately CP is to SQ as PM to $N\beta$: but PM having been proved equal to CY, and that $G\gamma$ is equal to $N\beta$; PM is to $N\beta$ as CY to $G\gamma$: and the triangles CVY, $GK\gamma$ being equiangular, CY is to $G\gamma$ as CV to GK; therefore CP is to SQ as CV to GK: hence the square of CP is to the square of SQ, as the square of CV to the square of GK: but the square of CP is to the square of CS as the square of

of

of CE to the square of CK; and, by conversion, and prop. 47. 1. Elem. the square of CP is to the square of SQ as the square of CE to the rectangle EKT: the square, therefore, of CE is to the rectangle EKT, as the square of CV to that of GK; alternately, the square of CE is to the square of CV as the rectangle EKT to the square of GK: therefore the square of ET is to that of VX, as the rectangle EKT to the square of GK.

Universally, then, the square of any diameter is to the square of its conjugate, as the rectangle contained by the segments, intercepted between its vertices and a straight line ordinately applied to it, is to the square of the segment (intercepted between the ellipse and it) of the straight line ordinately applied: for an ordinate to a diameter is parallel to the tangent drawn through the vertex of that diameter; and therefore is parallel to the diameter conjugate to that diameter.

COR. I. The squares of straight lines ordinately applied to the same diameter, are to one another as the rectangles contained by the segments of that diameter, as was demonstrated

demonstrated [1. cor. 6. 2.] with regard to the axes.

COR. 2. If ET , VX be conjugate diameters of an ellipse AT , and from a point G a straight line GK be drawn parallel to VX , one of the conjugates, and meeting the other ET in K ; and if the square of ET be to the square of VX as the rectangle EKT to the square of GK ; the point G is in the ellipse: for if GK meet not the ellipse in the point G , on that side, on which G is, of the diameter ET ; let it meet the ellipse in δ : then, by the proposition, the rectangle EKT is to the square of δK as the square of ET to the square of VX , that is, by hypothesis, as the rectangle EKT to the square of GK : hence the square of δK is equal to the square of GK ; and thus the straight line δK is equal to the straight line GK ; which is impossible.

COR. 3. If from two points G , ϵ , one of which, ϵ , is in the ellipse, there be drawn to the diameter ET straight lines GK , $\epsilon\zeta$ parallel to straight lines ordinately applied to the same ET ; if the rectangles EKT , $E\zeta T$, contained by the segments of the diameter ET , which are intercepted between its vertices,

tices and the parallels drawn to it, have the same ratio to each other as the squares of GK, $\&$; the other point G is likewise in the ellipse. This is demonstrated from cor. 1. in the same manner as the second corollary from the proposition.

COR. 4. If a circle be described upon AB, a diameter of the ellipse, and straight lines DE, FG be drawn ordinately applied to the diameter AB; and from the points D, F straight lines DH, FK be drawn perpendicular to the same AB, and meeting the circle in the points H, K; the perpendiculars DH, FK have the same ratio to each other which the ordinates DE, FG have; for the squares of DE, FG have the same ratio to each other which the rectangles ADB, AFB have, that is, which the squares [35. 3. Elem.] of DH, FK have: therefore DH is to FK as DE to FG, [22. 6. Elem.] Fig. 16.

COR. 5. And if two straight lines ordinately applied to a diameter, cut off, between the centre and the points where they meet that diameter, equal segments of that diameter, they are equal: and if equal, they cut off, between the centre and these points, equal segments.

O

PROP.

PROP. XVI. THEOR.

If from a point of an ellipse a straight line be ordinately applied to a diameter, and from the point where the ordinate meets the diameter a perpendicular be drawn meeting the circumference of a circle described upon the diameter; if a straight line touching the circle in the extremity of the perpendicular meet the diameter; the straight line that joins the point where that tangent and the diameter meet, and the point in the ellipse, touches the ellipse. On the other hand, if the straight line touching the ellipse meet the diameter, the straight line that joins the point where they meet, and the extremity of the perpendicular, touches the circle in that extremity.

E

E being a point in the ellipse, ED drawn ordi-
nately applied to AB a diameter, and there being drawn from the point D, where the ordinate and diameter meet, a perpendicular DH, that meets the circle described upon AB in H; if a straight line touching the circle in H meet the diameter AB in the point L; the straight line EL that joins L and the point E in the ellipse, touches the ellipse in the point E. Fig. 16.

For if EL touch not the ellipse, let it, if possible, meet it in another point M; and through M to the diameter AB draw MN parallel to ED; and let NO, drawn through N at right angles to the diameter AB, meet the circle in O on the same side of AB on which H is: since, then, the parallels DE, NM are drawn from the points D, N, as also the parallels DH, NO, which have the same ratio to each other [4 cor. preced.] which DE, NM have, and that the points E, M, L are in a straight line; therefore the points H, O, L [lem. 2.] are in a straight line; the straight line LH, therefore, cuts the circle; but, according to the hypothesis, it touches the circle; which is absurd: therefore LE touches the ellipse.

It may be shewn, after the same manner, that if EL touches the ellipse, LH touches the circle.

COR. Hence a method is pointed out, by which if a diameter of an ellipse be given in position and magnitude, and the angle which that diameter makes with any straight line ordinately applied to it, be given, a straight line can be drawn that will touch the ellipse in a given point.

P R O P. XVII. THEOR.

If from a point of an ellipse a straight line be drawn touching that ellipse, and meeting a diameter; if from the point of contact a straight line be ordinately applied to that diameter; the semidiameter is a mean proportional between the segments of the diameter that are intercepted between the centre and the tangent, and between the centre and the ordinate; and the segments

ments (of the diameter) intercepted between its vertices and the tangent, have the same ratio to each other as the segments (of the same diameter) intercepted between its vertices and the ordinate.

E being a point in the ellipse, EL touching the ellipse, and meeting the diameter AB in L, and there being drawn from the point of contact a straight line ED ordinately applied to the diameter AB; the semidiameter CB is a mean proportional between CL, CD, the segments (of the diameter) intercepted between the centre and the tangent, and between the centre and the ordinate. Fig. 16.

Having described a circle upon the diameter AB, and drawn from the point D a straight line DH at right angles to AB, and meeting the circle in H, join HL: then, because LH touches the [16. 2.] circle, and that HD is perpendicular to the diameter, CD, CB, CL are proportionals, [8. 6. Elem.]

Next, because CL is to CB as CB to CD,
by

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by conversion CL is to BL as CB to BD: double the antecedents, and twice CL is to BL as AB to BD; and, by division, AL is to BL as AD to BD.

COR. 1. Hence the rectangle contained by the segments (of the diameter AB) intercepted between the ordinate and the centre, and between the ordinate and the tangent, is equal to that contained by the segments between the ordinate and the vertices of the diameter: for as CD, CB, CL are proportionals, the square of CB is equal to the rectangle DCL; but the square of CB is equal to the rectangle ADB, together with the square of CD, [5. 2. Elem.]; and the rectangle DCL is equal to the rectangle CDL, together with the same square of CD, [3. 2. Elem.]: therefore the rectangle ADB, and the square of CD, are together equal to the rectangle CDL added to the same square of CD: take away the common square of CD, and there remains the rectangle ADB equal to CDL.

COR. 2. And the rectangle contained by the segments (of the diameter AB) intercepted between the tangent and the centre, and

and between the tangent and the ordinate, is equal to that contained by the segments between the tangent and the vertices of the same diameter AB: for the rectangle ALB and square of CB are together equal [6. 2. Elem.] to the square of CL; and the rectangles LCD, CLD are together equal [2. 2. Elem.] to the same square of CL: therefore, because the square of CB has been proved equal to the rectangle DCL, there remains the rectangle ALB equal to the rectangle CLD.

PROP. XVIII. THEOR.

From a point of an ellipse let a straight line be ordinately applied to a diameter, and from the same point let a straight line be drawn meeting the diameter on the same side (of the ordinate) on which the nearer (to the ordinate) of the vertices of the diameter is situated: if the segment (of the diameter) intercepted between the centre and the

the point where the diameter and the straight line meet, and the semidiameter, and the segment between the centre and the ordinate be proportionals; if, next, the segments between the point of concurrence of the diameter and straight line, and the vertices of the diameter, and between the ordinate and those vertices, be proportionals; if, thirdly, the segments between the ordinate and point of concurrence of the diameter and straight line, between the ordinate and vertices of the diameter, and between the ordinate and centre, be proportionals; or, last of all, if the segments between the point of concurrence of the diameter and straight line, and the more remote (from the ordinate) of the vertices

vertices of the diameter, between the same point of concourse and the centre, between the same point of concourse and the ordinate, and between the same point of concourse and the other vertex of the diameter, be proportionals: in all these cases, the straight line drawn from the point in the ellipse to the diameter produced, touches the ellipse.

Let E be the point in the ellipse, and ED an ordinate to the diameter AB ; and let EL , drawn from the point in the ellipse, meet the diameter in L : if the segment (of the diameter) CL , intercepted between the centre and point of concourse L of the diameter and EL , the semidiameter CB , and the segment DC , between the ordinate and centre, be proportionals; next, if the segments AL , BL , between the point L and the vertices of the diameter, have the same ratio to each other, as the segments AD , BD , between the ordinate and

P

vertices;

Fig. 16.

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vertices; if, thirdly, the segments DL , DA , DB , DC , between the ordinate and each of the points L , A , B , C , be proportionals; or, last of all, if the segments AL , CL , DL , BL , between the point L and each of the points A , C , D , B be proportionals: in all these cases the straight line EL touches the ellipse.

Case 1. If EL does not touch the ellipse, let EP touch it: therefore, by the preceding proposition, CP , CB , CD are proportionals: but, by hypothesis, CL , CB , CD are likewise proportionals; CP is, therefore, equal to CL ; which is absurd: consequently EL touches the ellipse.

Case 2. Because, by hypothesis, AL is to BL as AD to BD , by composition AL and BL together are to BL as AB to BD : take the halves of the antecedents, and CL is to BL as CB to BD ; and, by conversion, CL is to CB as CB to CD : therefore, by the first case, EL touches the ellipse.

Case 3. Since DL is to DA as DB to DC , by composition LA is to AD as BC or CA to CD ; the remainder CL is, therefore, to the remainder AC or CB , as CB to CD ; and, therefore, by the first case, again, EL touches the ellipse.

Case

Case 4. By hypothesis, AL is to CL as DL to BL ; therefore, by division, AC or CB is to CL as DB to BL ; CB , therefore, is to CL as the remainder CD to the remainder CB ; and, inversely, CL is to CB as CB to CD : therefore EL touches the ellipse.

PROP. XIX. THEOR.

If from two vertices of two conjugate diameters two straight lines be ordinately applied to another diameter, the square of the segment (of that other diameter) intercepted between either ordinate and the centre, is equal to the rectangle contained by the segments between the other ordinate and the vertices of that same diameter.

Let CA , CB be the two conjugate diameters, of which the points A , B are vertices; from A , B let AF , BG be ordinately applied to another diameter DE ; the

Fig. 171

P 2

square

square of CG , intercepted between the ordinate BG and the centre, is equal to the rectangle EFD contained by the segments intercepted between the other ordinate AF and the vertices of DE ; and the square of CF , in like manner, is equal to the rectangle EGD .

Draw AH , BK touching the ellipse in A , B , and meeting the diameter ED in H , K : then, because both CB , AH , and BG , AF , are parallel, the triangle CBG is equiangular to HAF ; and because BK is parallel to CA , the triangle CBK is also equiangular to HAC ; therefore CG is to FH as CB to AH , that is, as CK to CH : but CD is a mean proportional both between CG , CK , and between CF , [16. 2.] CH ; therefore CF is to CG as CK to CH : and CG , as hath been shewn, is to FH as CK to CH ; therefore as CF to CG , so is CG to FH : and, consequently, the square of CG is equal to the rectangle CFH : but the rectangle EFD is equal to the same [1. cor. 17. 2.] CFH ; hence the square of CG is equal to the rectangle EFD ; and take these equals from the square of CD , and there will remain the rectangle EGD equal to the square of CF , [see prop. 5. 2. Elem.]

COR.

COR. 1. Hence the femidiameter CD, to which the ordinates are drawn, is to the femidiameter conjugated to it as the distance between either ordinate and the centre is to the other ordinate: for the square of CD is to the square of CL as the rectangle EFD to the square of AF, that is, (by the proposition), as the square of CG to the square of AF; therefore CD is to CL as CG to AF. In like manner, it may be shewn, that CD is to CL as CF to BG.

COR. 2. The squares of the segments of the diameter to which the ordinates are drawn, which segments are between the ordinates and the centre, are together equal to the square of the femidiameter: for since the square of CG is equal to the rectangle EFD, the square of CF, together with the square of CG, is equal to the square of CF, together with the rectangle EFD, that is, to the [5. 2. Elem.] square of CD.

COR. 3. Hence the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes. Let CD, CL be the semiaxes, and CA, CB conjugate femidiameters; let AF, BG be perpendiculars to CD, and AM, BN perpendiculars

culars to CL: then, because the square of CD, as was proved in the preceding corollary, is equal to the square of CF, together with the square of CG, and that, by the same corollary, the square of CL is equal to the square of CM, together with that of CN, that is, to the square of AF, together with that of BG; the squares of CD, CL are together equal to the squares of CF, CG, AF, BG: but the squares of AC, BC are together (47. 1. Elem.) equal to the same squares of CF, CG, AF, BG; therefore the squares of AC, BC are together equal to the square of CD added to the square of CL: Hence the sum of the squares of two conjugate diameters is equal to the sum of the squares of the axes.

PROP. XX. THEOR.

If through the vertices of two conjugate diameters four straight lines be drawn touching the ellipse; the parallelogram formed by these straight lines, is equal to that formed by the tangents drawn through the vertices of any

any other two conjugate diameters.

Let the straight lines OV, OY, YX, XV touch the ellipse in the vertices A, B, and in the vertices opposite to A, B of two conjugate diameters AC, BC; in like manner, let PT, PR, RS, ST touch the ellipse in the vertices of the conjugate diameters DC, LC; the figures OVXY, PRST are [6. cor. 10. 2.] parallelograms, and equal to each other. Fig. 17.

To the diameter CD draw AF, BG parallel to CL, and to the diameter CL draw AM, BN parallel to CD; and let AO, BO meet the same CD in the points H, K; and having joined BH, complete the parallelogram HCNQ:

Then, because AH touches the ellipse, and that AF is drawn ordinately applied to the diameter CD, CH is to CD as CD to CF: and, by the first corollary of the foregoing proposition, CD is to CL as CF to BG: therefore, by an inference, founded on what is called *equality of distance*, [see 19. def. B. 5. Elem.] CH is to CL as CD to BG or QH; and the angles DCL, CHQ are equal, for they are alternate; therefore the parallelogram DL is equal to the parallelogram

gram [14. 6. Elem.] NH ; but NH is the double of the triangle CBH , upon the same base, and between the same parallels; and the parallelogram $ACBO$ is the double of the same triangle CBH , upon the same base CB , and between the same parallels CB , AH ; $ACBO$, therefore, is equal to NH ; and the parallelogram DL , as hath been shewn, is equal to the same NH ; therefore the parallelograms DL , AB are equal; and thus the parallelograms $PRST$, $OVXY$, which are the quadruples of DL , AB , are equal.

PROP. XXI. THEOR.

If a straight line touching an ellipse meet two conjugate diameters, the rectangle contained by its segments, between the point of contact and the diameters, is equal to the square of the semi-diameter conjugated to that which passes through the point of contact.

2

Let

Let the straight line HZ touch the ellipse in the point A, and let it meet the conjugate diameters CD, CL in H, Z; let the femidiameter CB be conjugated to CA; the rectangle HAZ is equal to the square of CB. Fig. 17.

For draw AF, BG parallel to the diameter CL: and since HF is to FC as HA to AZ, the rectangles HFC, HAZ are similar; and HF is to HA as CG is to CB; therefore, since the rectangles HFC, HAZ are similar, of which HF, HA are homologous sides, and that the squares of CG, CB are similar figures, the rectangle HFC [22. 6. Elem.] is to the rectangle HAZ as the square of CG to the square of CB: but the rectangle HFC is equal to the rectangle [1 cor. 17. 2.] EFD, that is, to the [19. 2.] square of CG; and therefore the rectangle HAZ is also equal to the square of CB.

COR. If a straight line HAZ touch an ellipse, and meet two conjugate diameters CH, CZ; if the rectangle HAZ be equal to the square of the femidiameter CB conjugated to CA, which passes through the point of contact; CH, CZ are two conjugate diameters.

Q

PROP.

PROP. XXII. THEOR.

If from a point of an ellipse a straight line be ordinately applied to a diameter, the rectangle contained by the segments of the diameter is to the square of the ordinate as the diameter is to its *latus rectum*.

Fig. 18.

Let F be a point in an ellipse; from F draw FG, ordinately to the diameter AB; the rectangle AGB is to the square of FG as the diameter AB to its *latus rectum*.

For let BH be equal to the *latus rectum*; and since the diameter AB, its conjugate DE, and *latus rectum* BH [9. def. 2.], are proportionals, AB is to BH [cor. 20. 6. Elem.] as the square of AB to the square of DE, that is, as the rectangle AGB to the square of FG [15. 2.].

PROP. XXIII. THEOR.

If from a point of an ellipse a straight line be ordinately applied

plied to a diameter, and from the vertex of the diameter a straight line be drawn at right angles to the diameter, and equal to its *latus rectum*; the square of the ordinate is equal to the rectangle applied to the *latus rectum*; which rectangle has for its breadth the abscissa between the ordinate and the vertex of the diameter, and is deficient by a figure similar, and similarly situated, to the figure contained by the diameter and the *latus rectum*.

Let F be a point in the ellipse; from F Fig. 18.
draw FG, an ordinate to the diameter AB;
from the vertex of AB draw a perpendicular BH, equal to the *latus rectum* of AB;
the square of FG is equal to the rectangle applied to BH; which rectangle has the abscissa BG for its breadth, and is deficient by a figure similar, and similarly situated,

Q²

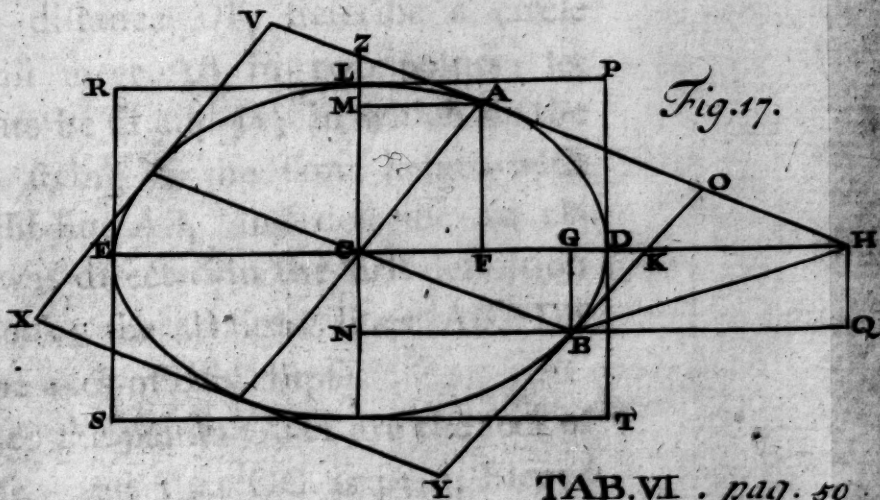
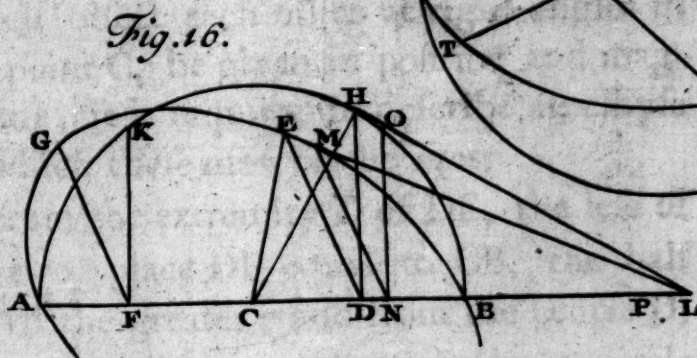
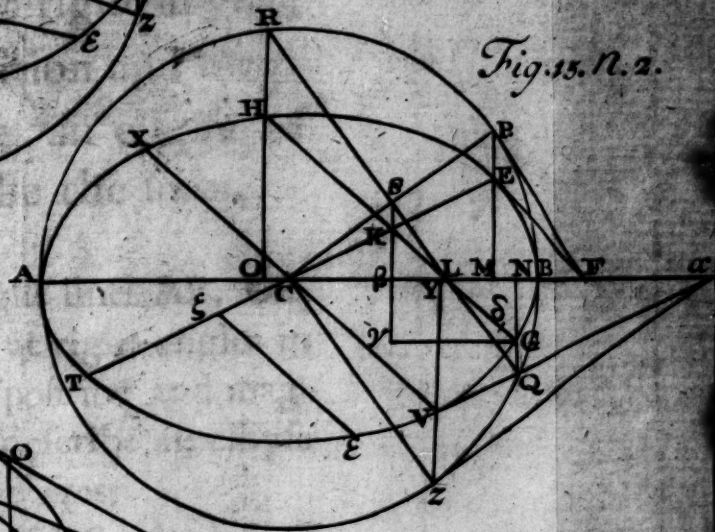
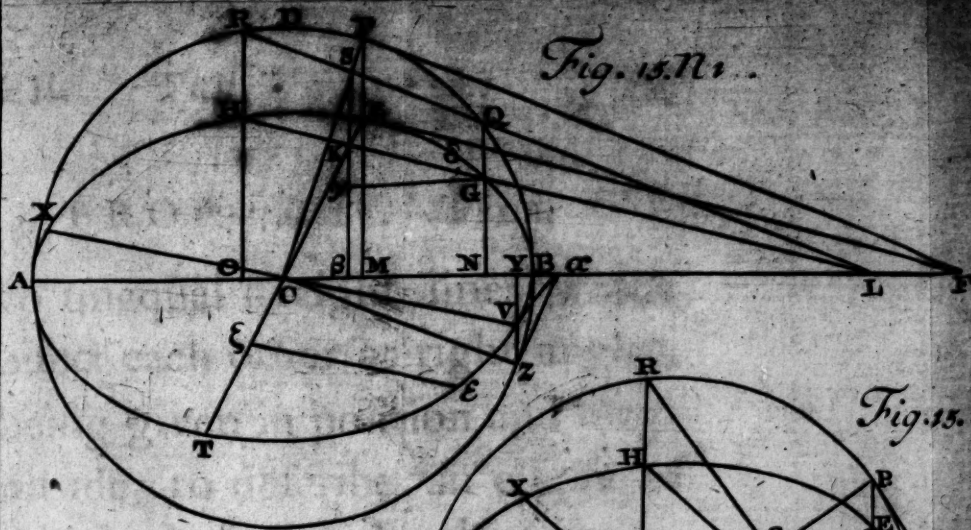
tuated, to the rectangle BN, contained by the diameter and *latus rectum* BH.

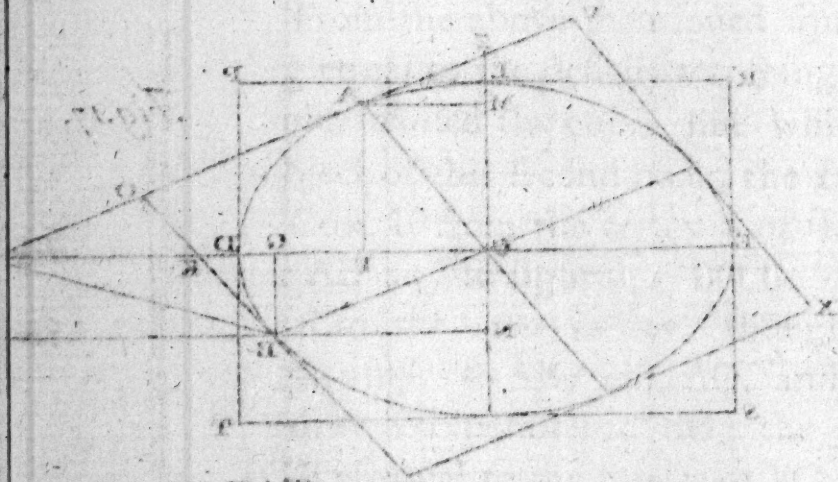
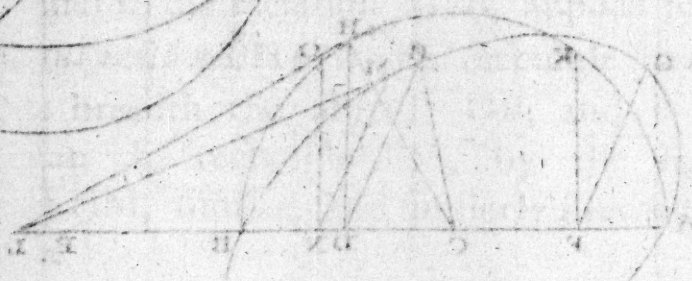
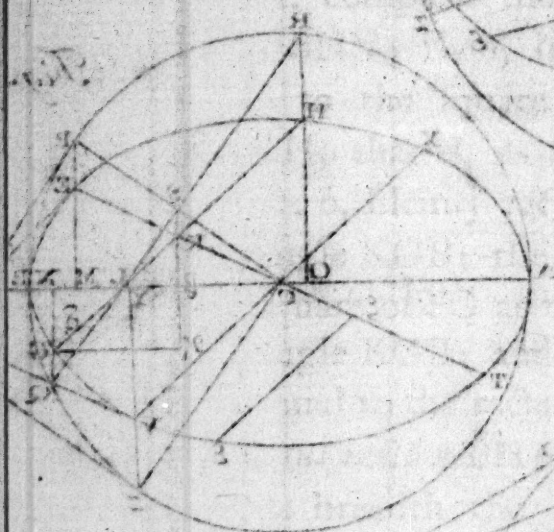
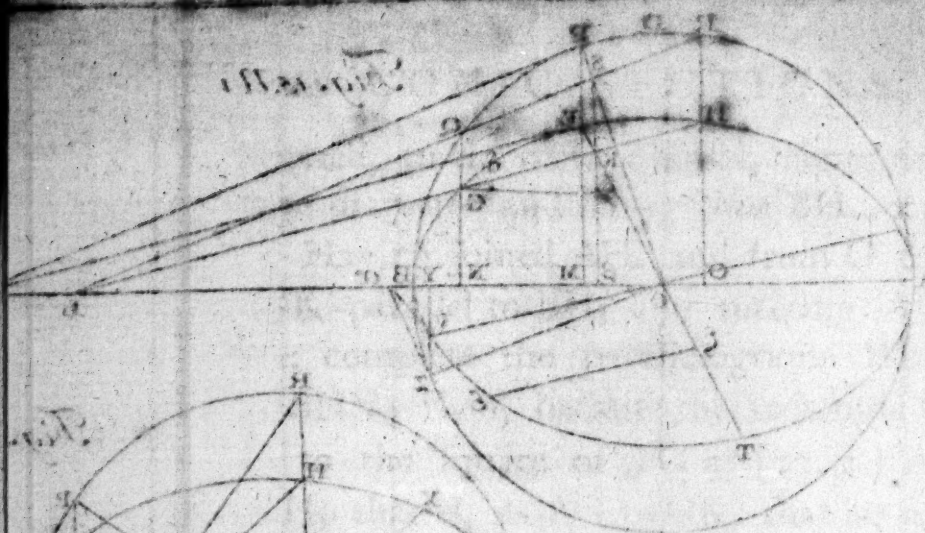
Having joined AH, and from G drawn GK parallel to BH, and meeting AH in K, complete the parallelograms KLHM, ABHN; then, because the rectangle AGB is to the square of FG as [22. 2.] AB to BH, that is, as AG to GK, that is, as the [1. 6. Elem.] rectangle AGB to the rectangle KGB; the rectangle AGB is to the square of FG as the same AGB to the rectangle KGB: and thus the square of FG is equal to the rectangle KGB, applied to the *latus rectum* BH; which rectangle has for its breadth the abscissa GB, and is less than the rectangle BM, by the figure KLHM, similar, and similarly situated, to BN.

From the above-mentioned square's being equal to the deficient rectangle, Apollonius named the curve line which is the subject of this second book, the *Ellipse*.

COR. If from the vertex B of the diameter AB any straight line BO be drawn equal to the *latus rectum*, though not at right angles to AB; join AO, and through G draw GP parallel to BO; the rectangle PGB is equal to the square of FG.

P R O P.





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PROP. XXIV. PROB.

Two unequal straight lines which bisect each other at right angles, being given in position and magnitude, to describe an ellipse of which these may be the axes.

Let two unequal straight lines AB, DE, which bisect each other at right angles in the point C, be given in position and magnitude; it is required to describe an ellipse of which these may be the axes.

Fig. 19.

From the extremity D of DE, the less of the two, place DF equal to CB, the half of AB the greater; and from the centre D, with the distance DF, describe a circle which will meet AB in two points; let these points be G and H; in which fix the ends of a string of the same length with the straight line AB, and describe an ellipse, as was directed in the first definition of this book; the straight lines AB, DE will be the axes of this ellipse.

For since the points G, H are the foci of the ellipse, and that GC is [3. 3. Elem.] equal

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equal to CH, the point C is its centre, [3. def. 2.]; and since CA or CB is equal to the length of the half of the string, the ellipse passes through the points A, B: it passes likewise through D [4. cor. 1. 2.], because GD is equal to CA; and through E, because CD is equal to CE.

PROP. XXV. PROB.

A straight line being given in position and magnitude, and a point without it being given; to describe an ellipse, of which that straight line shall be one of the axes, and which shall pass through that given point; but the given point must be so situated, that a perpendicular drawn from it may fall between the extremities of the given straight line.

Fig. 19. Let AB be the straight line given in position and magnitude, and K the given point, so situated, that a perpendicular drawn

drawn from it towards AB may fall between the extremities of AB; it is required to describe an ellipse which may have AB for one of the axes, and which may pass through K.

Draw KL at right angles to AB, and find a straight line DE such *, that the square of AB may be to the square of DE as the rectangle ALB to the square of KL; and place DE perpendicular to AB, and let them bisect each other, and with the same AB, DE, as the axes, describe an ellipse; this ellipse will pass through [2 cor. 15. 2.] the point K.

PROP. XXVI. PROB.

To find a diameter, the centre, the axes, and the foci of an ellipse given in position.

Draw two straight lines parallel to each other, and let them be terminated both

* Find a mean proportional between AL, LB [13. and 17. 6. Elem.]; then to the three following straight lines, viz. the mean proportional found, KL, and AB, find a fourth proportional [12. 6. Elem.]; this fourth proportional will be the DE wanted [22. 6. Elem.]

ways

ways by the ellipse; the straight line which bisects them is [4. cor. 13. 2.] a diameter, and the point bisecting that diameter is [3. 2.] the centre.

Fig. 20.

In order to find the axes, find the centre C, and in the ellipse take any point A, and join CA; and from the centre C, and with the distance AC, describe a circle AF: if this circle falls wholly without the ellipse, CA is the greatest of the semidiameters; and therefore the half of the [9. 2.] greater axis. Next, take any point D, and let a circle be described from the centre C, with the distance CD; if this circle falls wholly within the ellipse, CD is the least of the semidiameters; and therefore the half of the less axis [9. 2.]: or let the point G be taken; if the circle described from the centre C, with the distance CG, falls neither wholly without nor wholly within the ellipse, the straight line CG is the half neither of the greater nor of the less axis; the circle, consequently, must meet the ellipse again: let it meet it in H; and having joined GH, bisect it in K; CK will be one of the axes, and a straight line drawn through the centre perpendicular to CK will be the other: for since GH is bisected

ected by the diameter CK, it is parallel to the tangent AL drawn through the vertex of CK; and CKG is a right angle: therefore CAL is also a right angle; and therefore CKA is [4. cor. 10. 2.] one of the axes. The foci are found as in prop. 24.

PROP. XXVII. PROB.

Two conjugate diameters of an ellipse being given in position and magnitude, to find the axes, and describe the ellipse.

Let AB, CD, be the given diameters; Fig. 21.
let them meet each other in the centre E. Suppose the problem solved, that is, let FG, HK be the axes to be found, and through A draw the straight line AL parallel to CD; AL will touch [3. cor. 14. 2.] the ellipse in A, and will be given [28. dat.] in position: let the same AL meet the axes in the points L, M; therefore the rectangle LAM is equal to the square of CE, the half of the diameter conjugated to [21. 2.] AB: but CE is given, and, consequently, its square is given; there-
R fore

fore the rectangle LAM is given: Let the rectangle EAN be equal to LAM; and because EA is given in position and magnitude, AN is also given in position and magnitude; and because the rectangles LAM, EAN are equal, the points L, E, M, N are [conv. 35. 3. Elem.] in the circumference of a circle: therefore, if EN is bisected in O, the centre of the circle will be in the straight line OP, which is at right angles to EN [cor. 1. 3. Elem.]: but LEM being a right angle, the centre of the circle is likewise in [cor. 5. 4. Elem.] the straight line LM: it is therefore in the point P, where OP, LM intersect each other: therefore the centre P is given, and the point E is given: hence the circle described from the centre P, with the distance PE, is given in [6. def. dat.] position; so likewise are the points L, M, where its circumference meets AB, a straight line given in position: therefore the axes EL, EM are given in position: draw AQ at right angles to the axis FG; and because AL touches the ellipse, EQ, EF, EL are [17. 2.] proportionals; and EQ, EL are given: therefore EF is given in magnitude. It may in like manner be proved, that EK is given

given in magnitude; therefore the axes FG, HK are given in position and magnitude: and an ellipse described through the point A, with the axis FG [25. 2.], will be an ellipse, in which AB, CD are two conjugate diameters.

The composition is as follows. Produce EA to N, so that the rectangle EAN may be equal to the square of CE: bisect EN in O, and draw OP at right angles to it; in the point P let OP meet the straight line AL, which is parallel to CE; and from the centre P, distance PE, describe a circle, and let AP meet its circumference in the points L, M: join EL, EM, and draw AQ at right angles to EL; and between EQ, EL find [13. 6. Elem.] a mean proportional EF, and make EG equal to EF: then, by the 25th proposition of this book, describe an ellipse, of which FG may be one of the axes, and which may pass through the point A: of this ellipse AB, CD are conjugate diameters. For since AQ is perpendicular to the axis FG, and EQ, EF, EL proportionals, AL touches the ellipse [18. 2.] in the point A; and because CD is parallel to the tangent AL, it is in the same position with the diameter con-

jugated to AB; and the angle LEM being in a semicircle is a right angle; consequently EM is the other axis, and so is conjugated to FG: hence the rectangle LAM is equal to the square of the semidiameter conjugated to AE [21. 2.]: but the same rectangle LAM is equal [35. 3. Elem.] to the rectangle EAN, that is, by the construction, to the square of CE; therefore CE is the semidiameter conjugated to AE; the ellipse therefore passes through C; and because ED is equal to EC, and EB equal to EA, it passes likewise through the points D, B. Hence AB, CD are conjugate diameters in the ellipse described.

P R O P. XXVIII. P R O B.

The position and magnitude of a diameter of an ellipse being given, and the position being given of a straight line which, from a given point in the ellipse, is ordinately applied to that diameter; to describe the ellipse.

Let

Let BA be the given diameter, to which Fig. 21.
RS, a straight line given in position, is ordi-
nately applied, from a given point R of
the ellipse.

Bisect AB in the point E, and draw
through E a straight line parallel to RS,
and in that parallel take equal straight
lines EC, ED, so that the rectangle ASB
may be to the square of RS as the square
of AE to the square of EC or ED; then,
by the preceding proposition, describe an
ellipse of which AB, CD may be conju-
gate diameters: this ellipse will pass
through the [2. cor. 15. 2.] point R, and
RS will be ordinally applied [4. cor. 14. 2.]
to the diameter AB.

PROP. XXIX. THEOR.

If a cone cut by a plane passing
through the axis, be cut also by
another plane, meeting both the
sides, by which the angle at the
vertex is contained, of the tri-
angle through the axis, but
neither parallel to the base of
the

the cone, nor subcontrarily situated; if that other plane, and the plane in which the base of the cone is situated, meet in the direction of a straight line perpendicular either to the base of the triangle through the axis, or to the straight line which is in the same straight line with that base; the line which is the common section of that other plane, and the conical surface, is an ellipse, which has for one of its diameters the common section of the triangle through the axis, and of that same other plane.

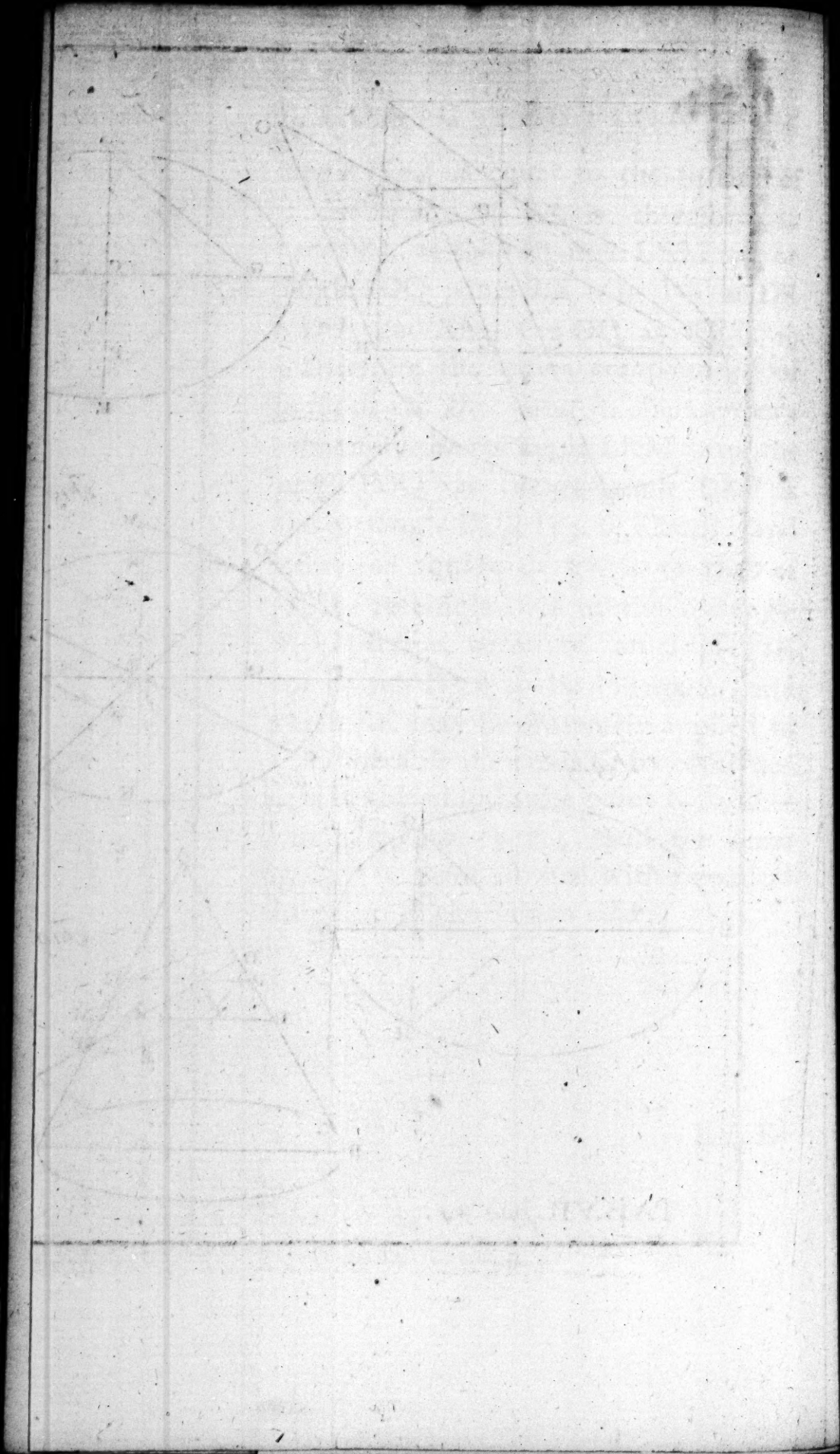
Fig. 22. Let there be a cone, the vertex of which is the point A, and the base the circle BC; let it be cut by a plane through the axis, and let the section be the triangle ABC; let it be cut likewise by another plane, meeting both the sides, by which the angle at A is contained, of the triangle through

through the axis, but neither parallel to the base of the cone, nor subcontrarily situated; let the line DEF be the common section of this other plane with the conical surface; and let the common section of this plane, with the plane in which is the base of the cone, be GH perpendicular to BC: the line DEF is an ellipse; and DF, the common section of the triangle through the axis, and this other plane, is one of its diameters.

In the section DEF take any point E, and through E to DF draw EK parallel to HG; next, through K draw LM parallel to BC: the plane, therefore, which passes through EK, LM is parallel to that through BC, GH, that is, to the base of [15. 11. Elem.] the cone: consequently the plane through EK, LM [23. 1.] is a circle, of which LM is a diameter: but EK is perpendicular [10. 11. Elem.] to LM, because GH is perpendicular to BG; the rectangle LKM is, therefore, equal [35. 3. Elem.] to the square of EK. In like manner, any other point N being taken in the section DEF, if NO be drawn parallel to EK, or GH, and through O, PQ be drawn parallel to BC, it may be shewn, that the
rectangle

rectangle POQ is equal to the square of NO: the square of EK is, therefore, to that of NO, as the rectangle LKM to the rectangle POQ: but LK is to PO as DK is to DO; and KM is to OQ as KF is to OF; therefore the ratios compounded of these ratios are the same to one another; and therefore the rectangle LKM is to the rectangle POQ as the rectangle DKF is to the rectangle DOF [23. 6. Elem.]: and therefore the square of EK is to that of NO as the rectangle DKF to the rectangle DOF. Describe, therefore, an ellipse [28. 2.] of which DF may be a diameter; and in which EK may be ordinately applied to DF: and because the point E, by construction, is in this ellipse, the point N is likewise in it [3. cor. 15. 2.]. And the same thing may be demonstrated with regard to all the points of the section DEF.

E L E.



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E L E M E N T S
OF THE
CONIC SECTIONS.

B O O K III.

Of the HYPERBOLA.

DEFINITIONS.

I. **I**N a point E taken in a plane, the ex- Fig. 1.
tremity E of a ruler EH is so fixed
that the ruler is left free to revolve about
the point as a centre: one end of a string
shorter than the ruler is fixed in the extre-
mity H of the ruler, and the other end
of it in the point F, which is in the same
plane with the point E; but the distance
between the points E, F must be greater
than the excess of the length of the ruler
above that of the string: by means of a
S pin

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pin G, as much of the string as possible is applied to the side EH of the ruler; then with the string so applied, and kept uniformly tense, the ruler is moved about the centre E: and thus by the point of the pin moving onwards with the ruler, a line is described, which is named the *hyperbola*.

But if the above order be reversed, and the end E of the ruler be fixed in the point F, and the end F of the string in the point E, and then a similar operation be repeated, another line, opposite to the former, will be described, which in like manner is named the *hyperbola*; and both together are named *opposite hyperbolas*. These lines may be extended beyond any given distance from the points E, F, if a string be taken, the length of which exceeds that distance.

II. The points E, F are named the *foci*.

III. And the point C, which bisects the straight line between the foci, is named the *centre of the hyperbola*, or of the *opposite hyperbolas*.

IV. Any straight line passing through the centre, and meeting the hyperbolas, is named a *transverse diameter*; and the points where a transverse diameter meets the
the

the hyperbolas, are named its *vertices*. Also a straight line which passes through the centre, and bisects any straight line terminated by the opposite hyperbolas, but not drawn through the centre, is named a *right diameter*.

V. The diameter which passes through the foci is named the *transverse axis*.

VI. If from either extremity A of the transverse axis, a straight line AD be placed equal to the distance between the centre C and either focus F, and from A as a centre, with the distance AD, a circle be described, meeting in the points B, *b* a straight line drawn through the centre C at right angles to the transverse axis; the straight line B*b* is named the *second axis*.

VII. Two diameters, each of which bisects all straight lines parallel to the other, and terminated both ways by the hyperbola, or opposite hyperbolas, are named *conjugate diameters*.

VIII. When a straight line not drawn through the centre, yet terminated both ways by the hyperbola, or opposite hyperbolas, is bisected by a diameter, it is said to be *ordinately applied* to that diameter, or it is named, simply, an *ordinate* to the diameter.

meter. Also a diameter parallel to a straight line ordinately applied to another diameter, is said to be *ordinately applied* to this other diameter.

IX. A straight line which meets the hyperbola in only one point, and which, being produced both ways, falls without the opposite hyperbolas, is said to *touch* the hyperbola in that point.

PROP. I. THEOR.

If from a point in an hyperbola two straight lines be drawn to the foci, the excess of the one above the other is equal to the transverse axis.

Fig. 1.

Let G be a point in an hyperbola, the excess of GE above GF is equal to the transverse axis Aa.

Let EGH represent the ruler, and FGH the string, the pin by which the hyperbola is described being supposed to remain at G; from EH, FGH take away the common part GH; and the excess of GE above GF will be equal to the excess of the length

length of the ruler above that of the string; and this conclusion will hold where-ever the point G shall be situated in the hyperbola. And since the points A, a , the vertices of the transverse axis, are in the opposite hyperbolas, the excess of AE above AF , and the excess of aF above aE , are each of them equal to the excess of the length of the ruler above that of the string, that is, to the excess of EG above GF ; and therefore these two excesses are equal to each other: but let AF be added to each of the two straight lines AE, AF ; and the excess of AE above AF will be equal to the excess of FE above twice AF . In like manner, the excess of aF above aE will be equal to the excess of the same FE above twice aE ; FE , therefore, exceeds twice AF by the same excess by which it exceeds twice aE : twice AF is, therefore, equal to twice aE ; and therefore AF is equal to aE : therefore the excess of AE above AF is equal to the excess of AE above aE , that is, to the transverse axis aA ; and therefore the excess of EG above GF is equal to the same transverse axis aA .

COR. And since AF is equal to aE and

CF

CF to CE, CA is equal to Ca, that is, the transverse axis is bisected in the centre.

PROP. II. THEOR.

Let there be a point from which two straight lines are drawn to the foci of opposite hyperbolas; if the excess of the one straight line above the other be equal to the transverse axis, that point is in one of the opposite hyperbolas.

Fig. 2.

Let G be a point, whence GE, GF are drawn to the foci; if the excess of the one straight line above the other be equal to the transverse axis aA , the point G is in one of the opposite hyperbolas.

Of the two straight lines let GF be the less; and from the centre F, distance FG, describe a circle; let this circle meet FE in H; take GK equal to GF, and KE, by hypothesis, will be equal to the transverse axis Aa : and because FG, GE are together greater than FE; FG, GK are together greater than FA and aE together; therefore

fore (FG, or) FH, the half of FG, GK, is greater than FA, the half of FA, aE ; and therefore the hyperbola, towards the point A, falls within the circle: and since it may be extended beyond any given distance from the focus F, it must cut the circle somewhere: Now it cuts the circle in the point G: For, if not, let it cut it in another point D, on the same side of the axis on which is the point G; and join DE, DF: the point D then is in the hyperbola; and therefore the excess of DE above DF is equal to the transverse axis Aa : but by hypothesis, the excess of GE above GF is equal to the same Aa ; and FG is equal to FD: therefore EG is also equal to ED; which is contrary to prop. 7. b. 1. of Euclid; therefore the point G is in the hyperbola.

PROP. III. THEOR.

If two straight lines be drawn from a point without an hyperbola to the foci, the excess of the one above the other is less than the transverse axis; but if
two

two straight lines be drawn from a point within an hyperbola to the foci, the excess of the one above the other is greater than the transverse axis. On the contrary, any point is without or within an hyperbola, according as the excess of two straight lines drawn from that point is less or greater than the transverse axis.

Fig. 2. From the point *L* without an hyperbola, let the two straight lines *LE*, *LF* be drawn to the foci; the excess of the one above the other is less than the transverse axis *Aa*: For since *L* is without and *F* within the hyperbola, the straight line *LF* must meet the hyperbola. Let *LF* meet it in *G*, and join *EG*, and *EL* will be less than *EG* added to *GL*; the excess of *EL* above *LF* is, therefore, less than the excess of *EG* and *GL* together above the same *LF*, that is, than the excess of *EG* above *GF*, that is, than the transverse axis *Aa*.

Next, from the point *M* within the hyperbola,

perbola, draw ME , MF to the foci; ME will necessarily meet the hyperbola, because the point M is within and the point E without it: let it meet it in N , and join NF ; then, because MF is less than MN added to NF , the excess of ME above MF is greater than the excess of the same ME above MN added to NF , that is, than the excess of NE above NF , that is, than the transverse axis Aa .

The last part of the proposition is obvious.

COR. Hence, if through the vertex A of the transverse axis, a straight line be drawn at right angles to the transverse axis, this straight line is wholly without the hyperbola, and consequently touches it.

For in the straight line take any point Q , and join QF , QE ; and in the axis place AR equal to AF , and join QR ; then, because AR is equal to AF , that is, to aE , RE is equal to the transverse axis Aa ; also QF is equal to QR : but QE is less than QR added to RE ; and therefore the excess of QE above QR or QF , is less than RE , that is, than the transverse axis: Hence the point Q , and consequently the straight line AQ , is without the hyperbola.

T

P R O P.

PROP. IV. THEOR.

The square of half the second axis is equal to the rectangle contained by the straight lines between either focus and the vertices of the transverse axis.

Fig. 1. Let Aa be the transverse axis, C the centre, E, F the foci, and Bb the second axis, which, from the definition of it, is bisected in the centre; join AB ; and because AB, CF are [6. def. 3.] equal, the squares of AC, CB are together equal to the square of CF , that is, to [6. 2. Elem.] the square of AC added to the rectangle AFa ; take away the common square of AC , and there will remain the square of CB equal to the rectangle AFa .

PROP. V. THEOR.

If from a point in an hyperbola a straight line be drawn at right angles to the transverse axis, and from that point a straight line be

be drawn to the nearer (to that point) of the foci; half the transverse axis is to the distance between that focus and the centre as the distance between the perpendicular and the centre is to the sum of half the transverse axis, and of the straight line drawn from the point to that same focus.

Let G be the point in the hyperbola; Fig. 3. 4. from G draw GD perpendicular to the transverse axis Aa ; and from the same point draw a straight line GF to the nearer (to G) of the foci; CA , half the transverse axis, is to CF , the distance between the centre and the focus F , as CD , the distance between the centre and the perpendicular, is to the sum of half the transverse axis and of GF .

Draw GF to the other focus, and in the axis aA produced place AH equal to GF , and from the centre C , distance GF , describe a circle, meeting the axis aA again in K , and the straight line EG in the points

L, M: and because EF is the double of CF, and FK the double of FD, EK is the double of CD. Again, because EL or Aa is the double of CA, and LM the double of GF or AH, EM is the double of CH: but, on account of the circle, EL or Aa is to EF as EK to EM [cor. 36. 3. and 16. 6. Elem.]; take the halves of these proportionals, and CA will be to CF as CD to CH.

PROP. VI. THEOR.

The same construction remaining, if from the nearer (to the point in the hyperbola) of the vertices of the transverse axis, and in this same axis produced, a straight line be placed equal to the distance between that point and the nearer (to that nearer vertex) of the foci; the square of the perpendicular is equal to the excess of the rectangle contained by the segments between the extremity

extremity (which is not the vertex of the transverse axis) of that straight line and the foci, above the rectangle contained by the segments between the perpendicular and the vertices of the transverse axis.

Let A be the nearer (to the point G) of Fig. 3.4. the vertices of the transverse axis; from A place in Aa produced AH equal to GF; the square of the perpendicular GD is equal to the excess of the rectangle EHF, contained by the segments between the extremity H of the placed line AH and the foci, above the rectangle ADa, contained by the segments between the perpendicular GD and the vertices A, a of the transverse axis.

For since the straight line CH is cut into any two parts in the point A, the squares of CA, CH are together equal to twice the rectangle ACH, together with the square of AH [7. 2. Elem.], that is (CA, CF, CD, CH being proportionals) to twice the rectangle FCD, together with the square of AH or GF, that is, to twice the rectangle

FCD,

FCD, together with the squares of FD, DG, that is, to the sum of the squares of FC, CD and DG [7. 2. Elem.]: therefore the sum of the squares of CA, CH is equal to the sum of the squares of FC, CD, DG: but the sum of the squares of CA, CH is equal [6. 2. Elem.] to the rectangle EHF, together with the sum of the squares of CA, CF; and the sum of the squares of FC, CD, DG is equal [6. 2. Elem.] to the rectangle ADa, together with the sum of the squares of CA, DG, and FC: hence the rectangle EHF, together with the sum of the squares of CA, CF, is equal to the rectangle ADa added to the sum of the squares of CA, DG, FC: from these equals take away the common squares of CA, CF, and there will remain the rectangle EHF, equal to the rectangle ADa, together with the square of DG.

PROP. VII. THEOR.

If from a point in an hyperbola a straight line be drawn parallel to the second axis, and meeting the transverse axis; the square
of

of the transverse axis is to the square of the second axis, as the rectangle contained by the segments (of the transverse) intercepted between its vertices and the parallel is to the square of the parallel.

Let G be the point in the hyperbola; from G draw GD parallel to the second axis Bb , and meeting the transverse in D ; the square of Aa is to that of Bb as the rectangle ADa , contained by the segments between the vertices of the transverse and the point D , is to the square of GD . Fig. 3. 4.

Having drawn two unequal straight lines GE , GF to the foci, place, from the nearer (to the focus F) of the vertices of the transverse axis, AH equal to GF the lesser of them: then, because CH , CD , CF , CA are proportionals, their squares are also proportionals; that is, as the whole, to wit, the square of CH , is to the whole the square of CD , so is the square of CF to the square of CA : therefore the remaining rectangle EHF is to the remaining rectangle ADa as the square of CF to that of CA [cor. 19. 5. Elem.]:

Elem.]: and, by division, the excess of the rectangle EHF above the rectangle ADa , is to ADa as the [6. 2. Elem.] rectangle AFa to the square of CA ; that is, by the preceding proposition, and fourth of this book, the square of GD is to the rectangle ADa as the square of CB is to that of CA ; and, inversely, the square of CA is to that of CB as the rectangle ADa is to the square of GD .

COR. The squares of straight lines drawn parallel to the second axis from points in an hyperbola, or in opposite hyperbolas, are to one another, as the rectangles contained by the segments intercepted between those straight lines and the vertices of the transverse axis, [see 1 cor. 6. 2.]

P R O P. VIII. THEOR.

If from a point in an hyperbola a straight line be drawn parallel to the transverse axis, and meeting the second axis; the square of the second axis is to that of the transverse

transverse, as the sum of the squares of half the second axis, and of its segment between the parallel and the centre, is to the square of the parallel.

From a point G of an hyperbola draw GN parallel to the transverse axis Aa , and meeting the second axis Bb in N ; the inference is as already expressed in the proposition. Fig. 4.

For, by the preceding, the square of CA is to the square of CB as the rectangle ADa is to the square of GD : therefore, inversely, and by proposition 12. B. 5. Elem. the square of CB is to the square of CA , as the sum of the squares of CB , GD is to the square of CA , together with the rectangle ADa , that is, as the sum of the squares of CB , CN is to the square of CD or GN .

COR. Hence, if from two points of an hyperbola, or of opposite hyperbolas, there be drawn to the second axis two straight lines parallel to the transverse axis, the square of the one straight line is to the

U square

square of the other, as the sum of the squares of half the second axis, and of the distance between the one straight line and the centre, is to the sum of the squares of half the second axis, and of the distance between the other straight line and the centre.

PROP. IX. THEOR.

A straight line terminated both ways by an hyperbola, or opposite hyperbolas, and parallel to either axis, is bisected by the other axis, or, what is the same thing, the axes are conjugate diameters.

Fig. 5.

First, let the straight line DE be parallel to the second axis Bb , and meet the transverse in F ; and thus the square of DF is to the square of EF as the rectangle AFa is to the [cor. 7. 3.] rectangle AFa ; hence DF , FE are equal.

Next, let DG be parallel to the transverse axis Aa , and meet the second axis Bb in K ; and thus the square of DK is to the

the square of KG , as the sum of the squares of CB , CK is to the sum of the same squares [cor. preced.] of CB , CK : hence DK , GK are equal.

PROP. X. THEOR.

A straight line terminated both ways by an hyperbola, or opposite hyperbolas, and bisected by either axis, is parallel to the other axis.

First, let DE be bisected by the transverse axis in F , and let DK , EL be drawn parallel to the same axis, and meeting the second axis in the points K , L ; then, because DF , FE are equal, KG , CL are also equal: but the square of DK is to that of EL as the square of CB , together with the square of CK , is to the square of CB , added to the square of CL ; DK , EL are, therefore, equal, and they are parallel: hence DE , KL are also parallel [3^d. 1. Elem.].

Fig. 5.

Next, let DG be bisected by the second axis in the point K , and let DF , GM be

drawn parallel to the same second axis, and meeting the transverse axis in F, M ; then, because DK, GK are equal, FC, CM are equal; and, of consequence, FA, aM are equal: but the square of DF is to the square of GM as the rectangle Afa to the rectangle AMa ; and the rectangles Afa, AMa are equal; therefore the straight lines DF, GM are equal, and they are parallel; therefore DG, FM are parallel [31. 1. Elem.].

COR. It is manifest from the demonstration, that the straight lines DF, GM , which are parallel to either axis Bb , and cut off, between the centre and the points where they meet the other axis, equal segments FC, MC , are equal. In the same manner, DK, EL are equal, which are parallel to the axis Aa , and cut off the equal segments CK, CL .

If, on the other hand, DF, GM are equal to each other, and parallel to Bb , they cut off equal segments FC, MC . In like manner, if DK, EL be equal to each other, and parallel to Aa , they cut off equal segments CK, CL .

P R O P.

PROP. XI. THEOR.

Any straight line perpendicular to the transverse axis, and meeting it below the vertex, meets the hyperbola in two points.

Perpendicular to the transverse axis AB , Fig. 6. 7.
and meeting it in C , below the vertex A ,
 DC meets the hyperbola in two points.

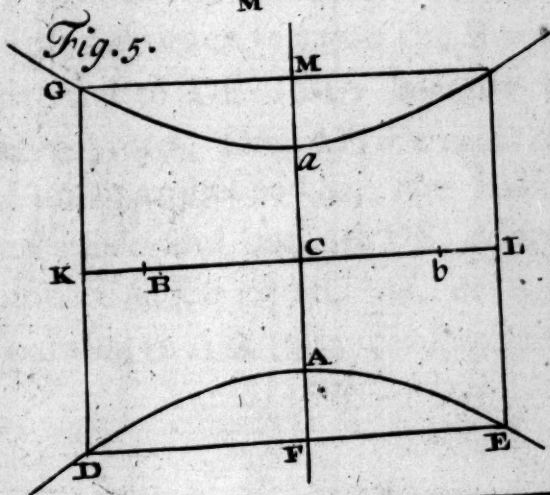
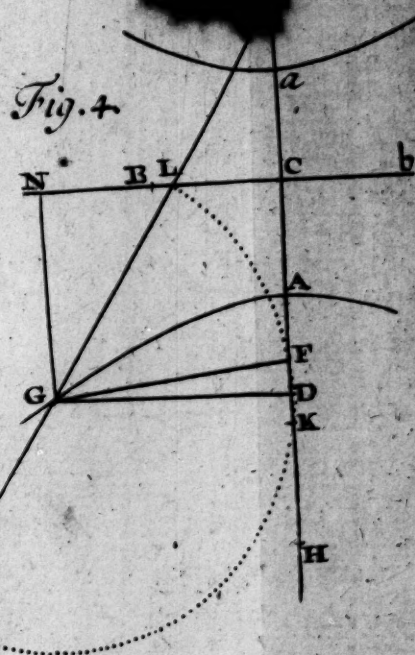
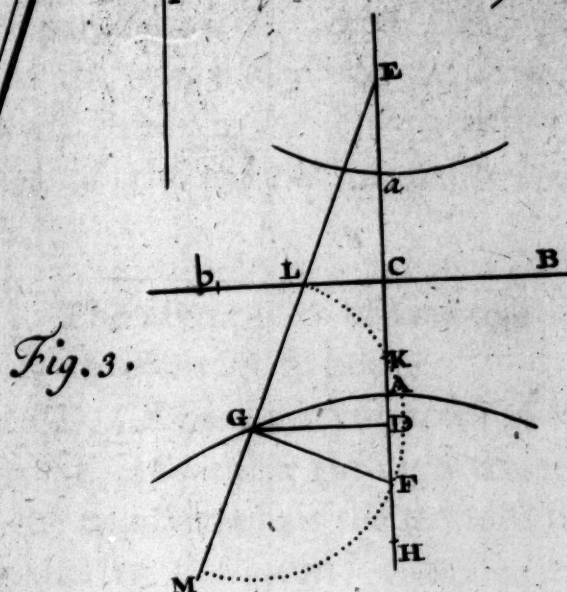
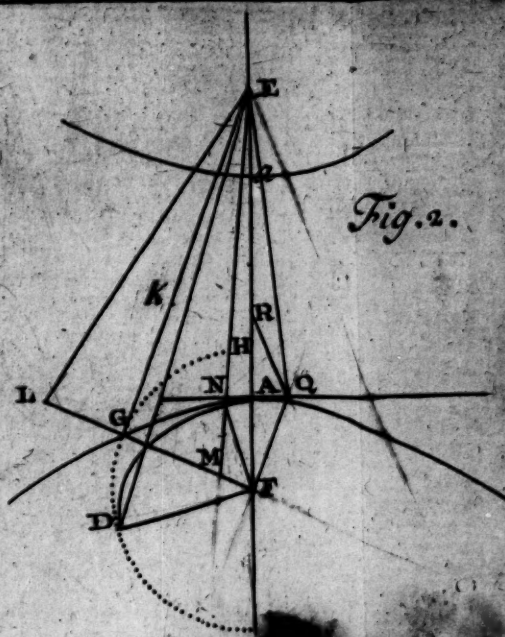
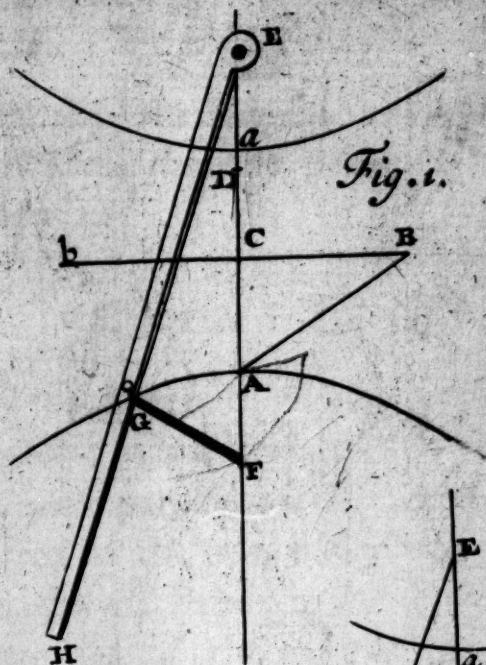
Let E, F be the foci ; and from C, place CG equal to CF, the distance between C and the nearer (to C) of the foci ; and from the other focus EK, equal to the transverse axis Aa. If, therefore, the point C be be- Fig. 6.

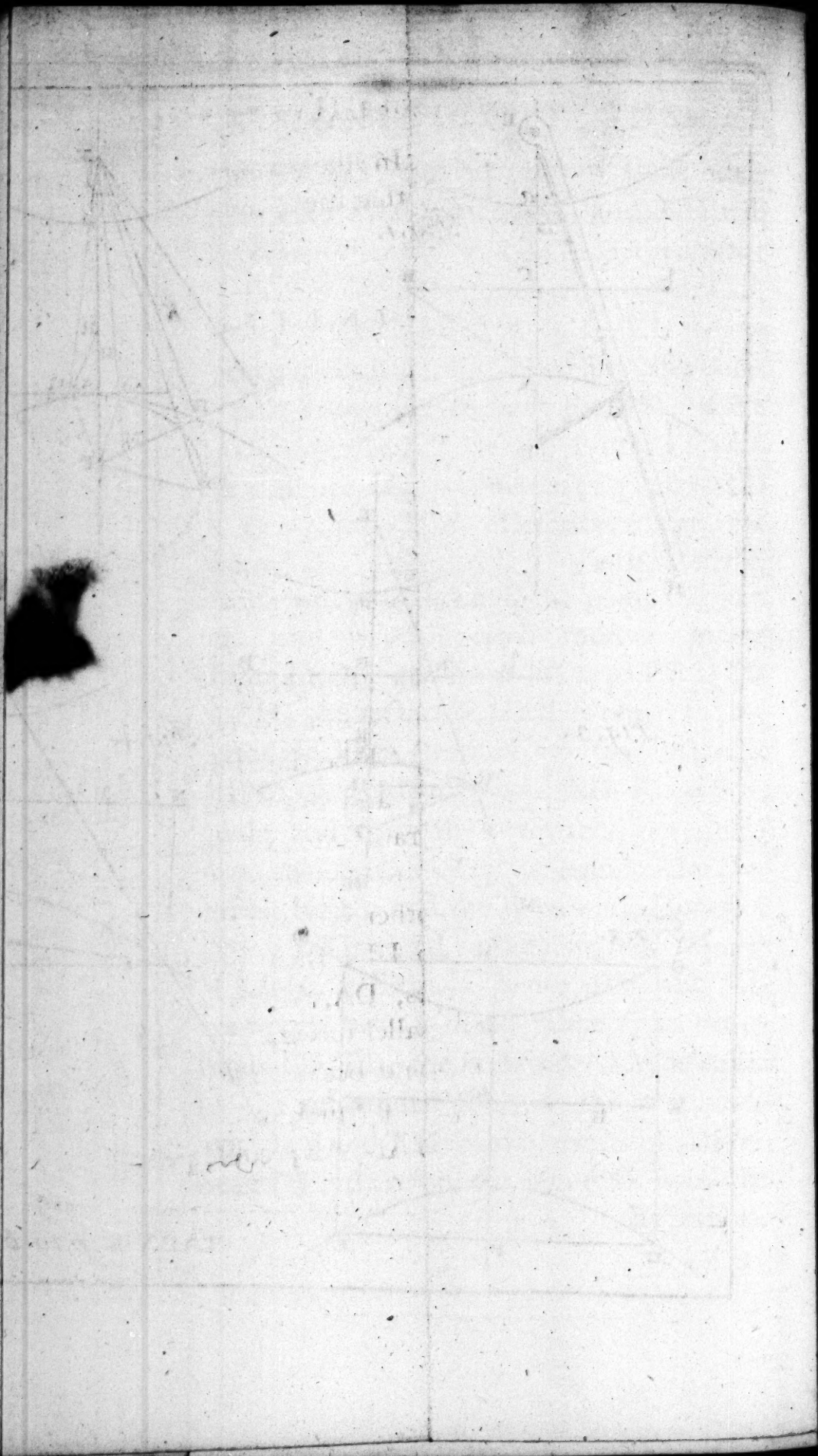
low the focus F , it is evident, that EK is less than EG : but in the case where the point C is above F , since Aa , EK are equal, Fig. 7.

AK is equal to aE , that is, to AF : and, by hypothesis, FC is less than FA ; twice FC is, therefore, less than twice FA , that is, FG is less than FK ; and thus EK is less than EG : make then EK to EF as EG is to a fourth proportional EH ; and since

EK is less than each of the two **EF**, **EG**,
and, of consequence, much less than **EH**,
EK, **EH** are [25. 5. Elem.] together greater
than

than EF, EG together. From these unequals take away twice EK, and KH will be greater than KF and KG together, that is, than twice KC; for CF is equal to CG: hence, if KH is bisected in L, KL will be greater than KC; and therefore the point L falls below the straight line CD; and a circle described from the centre E, with the distance EL, will necessarily meet CD in two points D, *d*. Describe, from the centre D, distance DE, another circle, which will pass through the point G; join DE, and let this circle meet it in the points M, N; and since EK is to EF as EG to EH, the rectangle HEK is equal to the rectangle FEG, that is, to the rectangle MEN [36. 3. Elem.]; and ED, EL are equal: and thus their squares are equal: from which take away the equal rectangles MEN, HEK, and the remaining square of DM or DF is equal to the remaining square of KL [6. 2. Elem.]: hence DM and KL are equal; which being taken from the equals ED, EL, the remainder EM is equal to the remainder EK, or the transverse axis Aa; and EM is the excess of DE above DF: therefore the point D is in the hyperbola.





hyperbola. In like manner it may be demonstrated, that the point d is in the hyperbola.

DEFINITION X.

If through one of the vertices of the transverse axis a straight line be drawn equal and parallel to the second axis, and bisected by the transverse axis; the straight lines drawn through the centre and the extremities of the parallel are named the *asymptotes*. Fig. 8.

COR. I. The asymptotes of two opposite hyperbolas are common to both.

For let CD , CE be the asymptotes of the hyperbola AF , and draw through the vertex A of the transverse axis the straight line DAE parallel to the second axis Bb , and through the other vertex a the straight line dae parallel to DE ; then because CD , CE are asymptotes, DA , AE are each of them equal and parallel to CB , the half of the second axis: and because DE , de are parallel, and CA , Ca equal, ad , ae are equal and parallel to AD , AE ; consequently they
are

are equal and parallel to half the second axis: therefore Cd , Ce are asymptotes of the opposite hyperbola a .

COR. 2. The asymptotes are parallel to straight lines joining the extremities of the axes; for if AB , bA be joined, CE , CD are parallel to them [33. 1. Elem.]

PROP. XII. THEOR.

The asymptotes do not meet the hyperbola.

Fig. 8. Let there be an hyperbola, the transverse axis of which is Aa , and the centre C , and through A draw a straight line perpendicular to CA ; and in this perpendicular take AD , AE , equal, each of them, to half the second axis; join CD , CE ; which are therefore the asymptotes; and, if possible, let CD meet the hyperbola in F , and through F draw a straight line parallel to DA , and meeting the axis Aa in G ; and since the rectangle AGa is to the square of GF , as the square of CA is [7. 3.] to that of CB or AD , that is, as the square of CG is to that of GF , the rectangle AGa is equal to the square of CG ; which is absurd:

furdi the asymptote, therefore, meets not the hyperbola in F; nor, as it may be shewn [6. 2. Elem.], does it meet the hyperbola in any other point.

PROP. XIII. THEOR.

If through a point of an hyperbola a straight line be drawn parallel to the second axis, and meeting the asymptotes; the rectangle contained by its segments intercepted between the asymptotes and that point, is equal to the square of half the second axis.

Let F be a point in the hyperbola; Fig. 8.
through F draw KFL parallel to the second axis, and meeting the asymptotes in the points K, L; the rectangle, &c.

Through the vertex A of the transverse axis draw DAE, meeting the asymptotes in the points D, E; and let KL meet the same axis in G: therefore AD, AE are each of them equal and parallel to half the second axis. To the second axis draw the

X

straight

straight line FM parallel to CA; and, by the 8th proposition of this book, the square of CB or AD will be to the square of CA as the sum of the squares of CB, CM is to the square of FM or GC; and the square of AD is to the square of AC as the square of KG to the square of GC; therefore the sum of the squares of CB, CM is to the square of GC as the square of KG is to the same square of GC: the sum of the squares of CB, CM is, therefore, equal to [9. 5. Elem.] the square of KG: from these equals take the equal squares of CM, FG, and the remaining square of CB is [5. 2. Elem.] equal to the remaining rectangle KFL. In like manner, if KL meets the hyperbola again in H, it may be shewn, that the rectangle KHL is equal to the square of CB.

COR. Hence if in a straight line KL terminated by the asymptotes, and parallel to the second axis, there be taken a point F, so situated, that the rectangle KFL may be equal to the square of the second axis; that point is in the hyperbola.

PROP.

PROP. XIV. THEOR.

If a straight line meeting an hyperbola in two points, meets also the asymptotes; the rectangle contained by the segments between the asymptotes and the one point is equal to that contained by the segments between the same asymptotes and the other point: Also if a straight line meeting two opposite hyperbolas in two points, meets likewise the asymptotes; the rectangle contained by the segments between the asymptotes and the one point, is equal to that contained by the segments between the same asymptotes and the other point: And in both cases the straight lines between the asymptotes and the points are equal.

Fig. 9. Let AB be a straight line meeting the hyperbola, or opposite hyperbolas, in the points A, B , and the asymptotes in C, D ; the rectangles CAD, CBD are equal; and CA, BD are equal.

Through the points A, B draw straight lines parallel to the second axis, and meeting the asymptotes in E, F and in G, H ; and since, by the preceding proposition, the rectangles EAF, GBH are each of them equal to the square of half the second axis, they are equal to each other; therefore, as EA to GB so is BH to AF : but the triangles being equiangular, as EA to GB so is CA to CB ; and as BH to AF so is BD to AD : therefore as CA to CB so is BD to AD : the rectangle, therefore, CAD is equal to the rectangle CBD : Take away* or add† the common rectangle AC, BD , and the rectangle CAB is equal to the rectangle ABD ; and therefore the straight lines AC, BD are equal [1. 6. Elem.].

COR. If from two points A, L in an hy-

* In the case where the points are in the same hyperbola.

† In the case where the points are in opposite hyperbolas.

perbola,

perbola, to either asymptote KC straight lines AM, LN be drawn parallel to the other asymptote; there being drawn to any point B in the hyperbola straight lines AB, LB, which meet the asymptote KC in C, O; CO, MN are equal: for let AB, LB meet the asymptote KP in the points D, P, and to the other asymptote KC draw BQ parallel to the asymptote KP; since AC, BD are equal, as also OL, BP; CM, QK are equal, as also ON, QK; CM, therefore, is equal to ON; and MO being common, CO, MN are equal.

PROP. XV. THEOR.

If through two points in an hyperbola, or in opposite hyperbolas, two parallel straight lines be drawn which meet the asymptotes; the rectangles contained by their segments between the points and the asymptotes are equal.

Let A, B be two points in an hyperbola, or in opposite hyperbolas; through these points

Fig. 10.
n. 1.

points draw CD , EF parallel to each other, and meeting the asymptotes OC , OD in the points C , D and E , F ; the rectangles CAD , EBF are equal.

Through the points A , B to the asymptotes draw the straight lines GAH , KBL parallel to the second axis; and because the rectangles GAH , KBL are each of them equal [13. 3.] to the square of half the second axis, they are equal to each other: therefore GA is to KB as BL to AH ; but the triangles GAC , KBE being equiangular, GA is to KB as CA to EB ; and the triangles LBF , HAD being equiangular, BL is to AH as BF to AD : hence CA is to EB as BF to AD ; and hence the rectangles CAD , EBF are equal.

Fig. 10.
D. 2.

COR. 1. And if through the centre a straight line AOM be drawn meeting both the hyperbolas, and parallel to the straight line BEF , the square of either segment AO , intercepted between the centre and either hyperbola, is equal to the rectangle EBF . The demonstration is the same as in the proposition.

COR. 2. Hence any straight line drawn through

through the centre, and terminated by opposite hyperbolas, is bisected in the centre.

COR. 3. If CD , EF meet an hyperbola, or its opposite hyperbola again in the points M , N , the rectangle ACM , or ADM , is equal to the rectangle BEN or BFN ; for AC , MD are equal, as also BF , NE . Fig. 10.
n. 1.

COR. 4. And since it has been proved, that the square of the semidiameter AO or OM is equal to the rectangle EBF , that is, to BEN ; BE , for this reason, is to AO as AO to EN ; and consequently BN is greater than [25. 5. Elem.] twice AO , that is, than AM ; that is, a transverse diameter is less than any straight line parallel to it, and terminated in opposite hyperbolas. Fig. 10.
n. 2.

COR. 5. If in a straight line BN terminated by the hyperbolas, there be taken points E , F , which make each of the rectangles BEN , BFN equal to the square of the semidiameter AO , which is parallel to BN ; the points E , F are in the asymptotes.

PROP.

PROP. XVI. THEOR.

If from a point in an hyperbola to the asymptotes any two straight lines be drawn, to which other two straight lines drawn to the asymptotes from any other point in the same or opposite hyperbola, are parallel; the rectangle contained by the former straight lines is equal to that contained by the latter.

Fig. 11.

Let A, B be points in an hyperbola, or in opposite hyperbolas; through A draw the straight lines AC, AD to the asymptotes, and through B draw BE, BF parallel to AC, AD; the rectangle CAD is equal to the rectangle EBF.

Draw through the points A, B to the asymptotes the straight lines GAH, KBL parallel to the second axis, and the proposition may be demonstrated in the same words with the preceding.

Fig. 6.

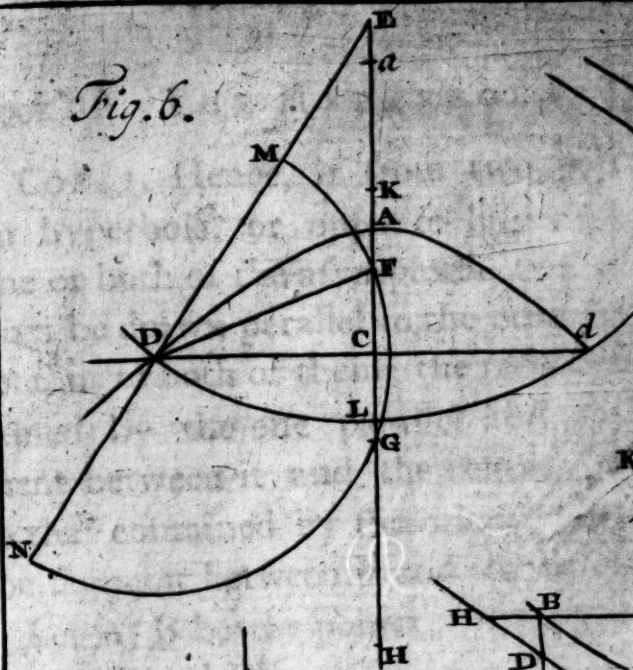


Fig. 8.

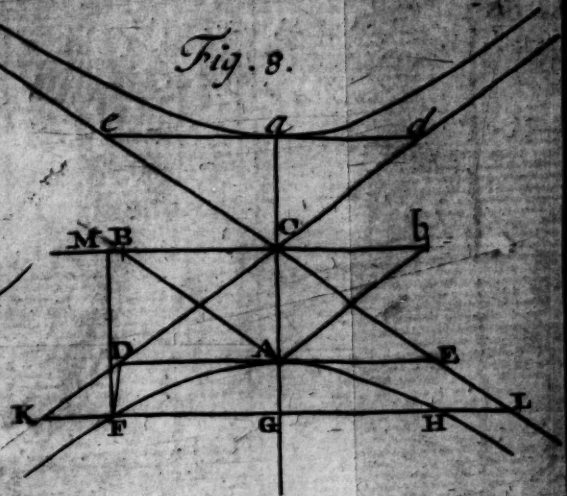


Fig. 9.

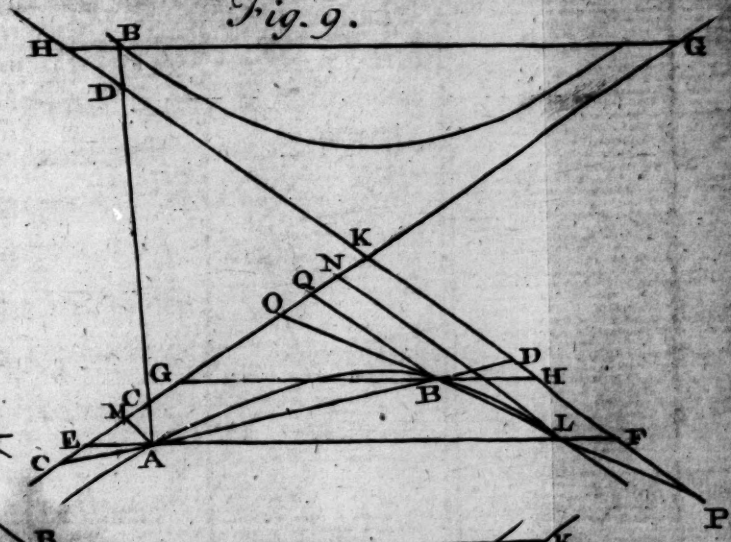


Fig. 7.

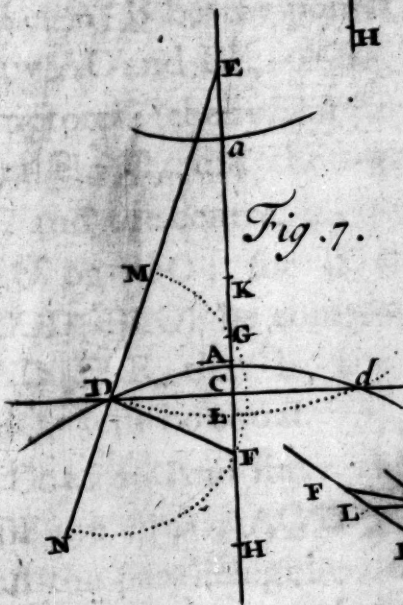
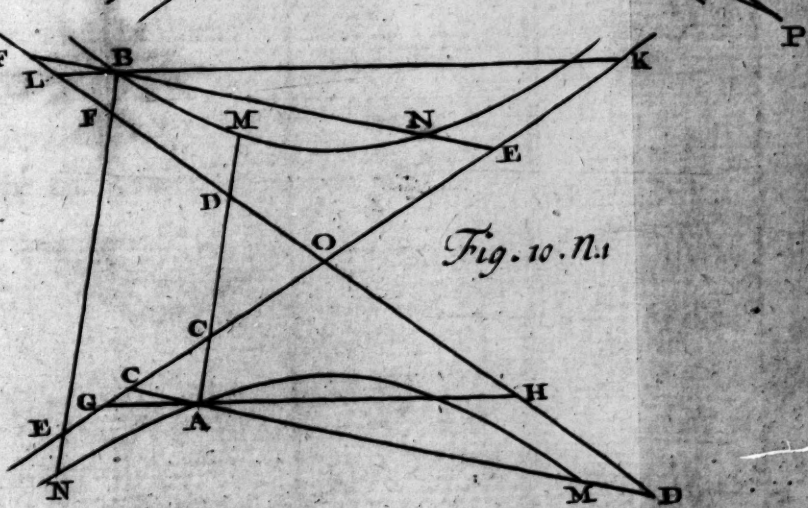
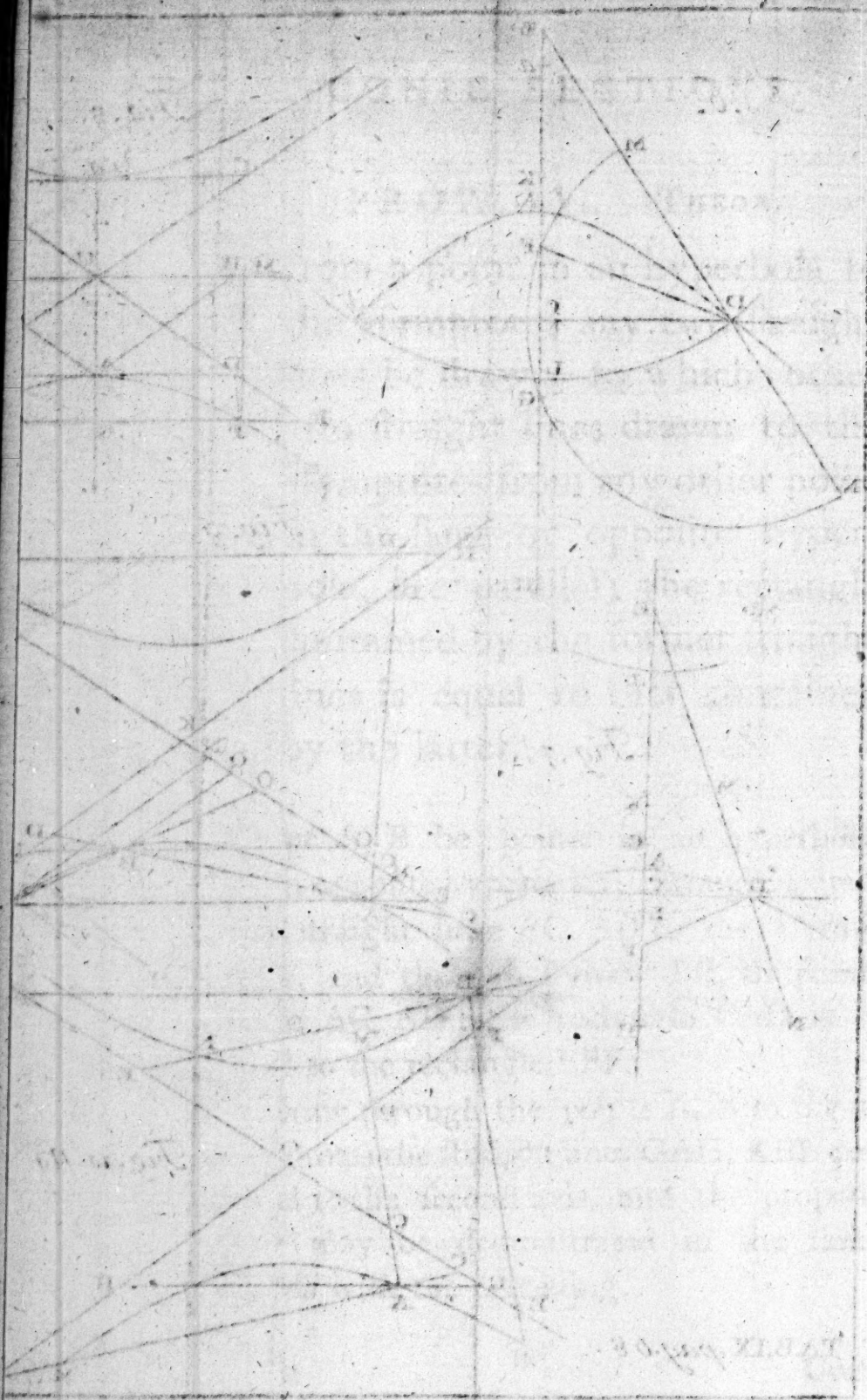


Fig. 10. N. 1.





COR. 1. Hence, if from two points in an hyperbola, or opposite hyperbolas, to one or both of the asymptotes, two straight lines be drawn parallel to the other asymptote, or to both of them; the rectangle contained by the one parallel and the segment between it and the centre, is equal to that contained by the other parallel and the segment between it and the centre.

Let A, B be the points; through them draw AC and BE, or BF, parallel to the asymptotes; the rectangle contained by the parallel AC, and the segment CO between AC and the centre, is equal to the rectangle BEO or BFO: for if the parallelograms ACOD, BEOF be completed, the rectangles CAD, EBF, that is, the rectangles ACO, BEO, will be equal.

COR. 2. And since the rectangles CAD, EBF are equal, AC is to BE as BF to AD; and the parallelograms ACOD, BEOF being equiangular, are therefore equal [14. 6. Elem.].

Y

~ ~ ~ P.

PROP. XVII. THEOR.

Any straight line drawn through the centre, and which passes within the angle contained by the asymptotes, meets the hyperbola.

Fig. 12.

Let AB , AC be the asymptotes, and AD the half of the transverse axis, and let AE be any straight line drawn through the centre and passing within the angle BAC ; AE meets the hyperbola. For if AE meets not the hyperbola, through D draw BDC parallel to the second axis, and meeting the asymptotes in B , C ; draw also DF parallel to AB , and let DF meet AE in F ; and having taken BG equal to DF , join GF , which will be equal and parallel to BD , and will therefore meet the transverse axis at right angles, and so will cut the hyperbola [11. 3.]: let it cut it in H on the same side of AD on which is the point F ; and let it meet the other asymptote in K : since, therefore, the point F is without the hyperbola, GH is greater than GF , that is, than BD ; and HK is greater than DC :
therefore

therefore the rectangle GHK is greater than the rectangle BDC, that is, than the square of BD: but, by the 13th of this book, the rectangle GHK is equal to the square of BD; which is absurd: therefore AE necessarily meets the hyperbola.

COR. If from the centre a straight line OA be drawn within the angle contained by the asymptotes, and the square of that straight line be equal to the rectangle EBF, contained by the segments of any straight line parallel to OA, which are intercepted between the point B, where that parallel meets the hyperbola, and the points E, F where it meets the asymptotes; the point A is in one of the hyperbolas: for, according to the proposition, the straight line OA necessarily meets the hyperbola: if, therefore, it meets not the hyperbola in A, let it meet it, if possible, in P; and then the square of OP will be equal to the [1 cor. 15. 3.] rectangle EBF, that is, to the square of OA; which is absurd: therefore the point A is in the hyperbola.

Fig. 10.
n. 2.

PROP. XVIII. [*Prop. 13. B. 2. Apoll.*]

If within the angle contained by the asymptotes any straight line be drawn parallel to either of the asymptotes, it meets the hyperbola in one point only, and passes within the hyperbola.

Fig. 12.

Let there be an hyperbola, the asymptotes of which are AL , AM ; take any point N , and through N draw NO parallel to AL ; NO will meet the hyperbola. For, if possible, let NO not meet the hyperbola; and in the hyperbola take any point P , through which draw PQ , PM parallel to AM , AL , and make the rectangle ANO equal to MPQ ; and having joined AO , produce it; AO will meet the [17. 3.] hyperbola: let it meet it in the point R , and through R draw RS , RT parallel to AM , AL ; therefore the rectangle MPQ is equal to the [16. 3.] rectangle TRS : but the rectangle ANO is made equal to the same MPQ ; the rectangle, therefore, TRS , that is, the rectangle ATR , is equal to ANO ;
which

which is impossible, because RT is greater than NO , and AF greater than AN : therefore NO must meet the hyperbola. Let it meet it in the point V , and it is to be proved that it does not meet it in any other point: For, if possible, let it meet it likewise in X , and through V, X draw VY, XL parallel to AM ; therefore the rectangle NVY is equal to the rectangle NXL ; which is absurd: therefore NO meets the hyperbola no where but in the point V . Lastly, in the straight line NV produced, take the point X , and through X draw a straight line parallel to AN , and let this parallel meet AY in L and the hyperbola in Z ; therefore the rectangle XLA is greater than VYA , that is, than ZLA ; therefore LX is greater than LZ ; and thus the point X is within the hyperbola.

COR. 1. It appears from the demonstration, that a straight line drawn through the centre, and passing between the asymptotes, meets the hyperbola in one point only: for should AR meet the hyperbola in another point O , the rectangles RTA, ONA would be equal; which is absurd.

COR. 2. And if a straight line meet an hyperbola,

hyperbola, or opposite hyperbolas, in two points, it meets both the asymptotes; for if it were parallel to the one of the asymptotes, it would meet the hyperbola in only one point.

COR. 3. And if a straight line touch an hyperbola it meets both the asymptotes; for if it were parallel to the one of them, it would pass within the hyperbola; which is absurd.

COR 4. If through the point N in one asymptote a straight line NO be drawn parallel to the other, and in this straight line, and within the angle within which the hyperbola is described, a point V be taken, making the rectangle VNA, contained by a straight line between the asymptote AM and the point V, and by the segment between that straight line and the centre, equal to the rectangle PMA, contained by a straight line drawn from any point P of the hyperbola, so as to be parallel to the asymptote AL, and by the segment between this parallel and the centre; the point V is in the hyperbola: for if NO meet not the hyperbola in V, let it, if possible, meet it in X; the rectangle XNA is, therefore, equal to the rectangle PMA, that

that is, to the rectangle VNA; which is absurd: therefore the point V is in the hyperbola.

PROP. XIX. THEOR.

If through a point A of an hyperbola a straight line be drawn meeting both the asymptotes in the points B, C; if from either of the points C another straight line CD be placed equal to the straight line intercepted between the point A in the hyperbola and the remaining point B, so that the extremity D of the straight line CD, and the point A in the hyperbola, may be either both between or both beyond the points B, C; the point D, in the one case, is in the hyperbola in which the point A is; but in the other, it is in the opposite hyperbola.

Fig. 13.

Let

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Let G be the centre of the hyperbolas, and through A, D , to either asymptote GB , draw straight lines AE, DF parallel to the remaining asymptote; and, because of the parallels, BA is to DC as BE to FG ; but BA, DC are equal; therefore BE, FG are equal; and so BF, EG are also equal: and because of the equiangular triangles AE is to DF as BE to BF , that is, as FG to EG the rectangle AEG is therefore equal to the rectangle DFG , and the point A is in the hyperbola; and therefore the point D (GF being one of the asymptotes) is also in the hyperbola [4 cor. preceding].

PROP. XX. THEOR.

If a straight line cuts both the straight lines containing the angle adjacent to that within which an hyperbola is described, it meets each of the hyperbolas in only one point.

Fig. 13.

Let there be an hyperbola, the asymptotes of which are GB, GC , and let the straight line BC cut them in the points

2

$B,$

B, C; and having taken in the hyperbolas any point H, through that point draw HIK parallel to BC, and meeting the asymptotes in I, K; and to the straight line BC apply a rectangle equal to the rectangle IHK, and let this applied rectangle exceed by a square [29. 6. Elem.]; and let A, D be the points of application; A, D are points in the hyperbolas. For through A, H draw the straight lines AE, AN, and HL, HM parallel to the asymptotes; and because the rectangles BAC, KHI are equal, BA is to KH as HI to AC: but, on account of the equiangular triangles, BA is to KH as AE to HL; and HI is to AC as HM to AN: therefore AE is to HL as HM to AN; and therefore the rectangle EAN, or AEG, is equal to the rectangle MHL, or HLG; and the point H is in the hyperbola; therefore the point A is also in the same or in the opposite hyperbola [4. cor. 18. 3.]; and in the same manner D is in the hyperbola opposite to the hyperbola in which the point A is: And it is manifest, that BC does not meet the hyperbolas in any other point.

COR. Hence if a straight line BC cut
Z both

both the straight lines containing the angle adjacent to that within which the hyperbola is described, and in BC produced a point A be taken, which makes the rectangle BAC equal either to KHI , contained by the segments of any straight line HK parallel to BC , which segments are intercepted between the point H where HK meets the hyperbola, and the points K, I where it meets the asymptotes; or to the square of the semidiameter parallel to BC ; the point A is in one of the hyperbolas.

PROP. XXI. THEOR.

Let a straight line cut both the straight lines within which an hyperbola is described; if the square of the half of it be not less than the rectangle contained by the segments of another straight line drawn parallel to it through any point of the hyperbola, which segments are between the hyperbola and the asymptotes;

asymptotes; that straight line meets the hyperbola.

Let there be an hyperbola, the asymptotes of which are OC, OD, and let a straight line GH cut them; GH meets the hyperbola, if the square of its half be not less than the rectangle mentioned in the proposition.

Fig. 11

Having taken in the hyperbola any point M, through that point draw a straight line parallel to GH, and meeting the asymptotes in K, L; and to the same GH apply a rectangle equal to the rectangle KML, and deficient by a square; which, from the determination, is possible [27. 28. 6. Elem.], and let A be the point of application; this point will be in the hyperbola: for if the straight lines AC, MN be drawn through the points A, M parallel to the asymptotes, the rectangles ACO, MNO will be equal; because the rectangle GAH is equal to the rectangle KML; and the point M is in the hyperbola: therefore the point A is also in it. In like manner it may be proved, that the other point of application is in the hyperbola: but if the square of the half of

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GH be equal to the rectangle KML, the point bisecting GH is the only point of GH that is in the hyperbola.

COR. Hence, if a straight line GH cut the asymptotes OK, OL of an hyperbola, and in GH a point A be taken, making the rectangle GAH equal either to the rectangle KML, contained by the segments of any straight line KL parallel to GH, which segments are intercepted between the point M where KL meets the hyperbola, and the points K, L, where it meets the asymptotes; or to the square of the segment of the tangent parallel to GH, which segment is between the asymptote and point of contact; the point A is in one of the hyperbolas.

PROP. XXII. [*Prop. 14. B. 2. Apoll.*]

An asymptote and the hyperbola, produced without end, continually * approach; and the distance between them becomes less than any given distance.

* See proposition 12.

Let

Let there be an hyperbola, the asymptotes of which are AB, AC , and let D be the given distance; and in the hyperbola let E, F be two points, through which draw GEH, CFL parallel to each other, and meeting the asymptotes in the points G, H and C, L ; join AE , and let it meet CL in K : then, because the rectangle GEH [15. 3.] is equal to the rectangle CFL , LF is to HE as EG is to FC : but LF is greater than HE , because KL is greater than HE ; therefore EG is also greater than FC . In like manner it may be proved, that the straight lines which follow are successively less than FC . Take then a distance GM less than the distance D , and through M draw MN parallel to AC ; therefore MN will meet [18. 3.] the hyperbola: Let it meet it in N , and through N draw ONB parallel to GH ; ON is, therefore, equal to GM , and therefore less than the distance D .

Fig. 14.

PROP.

PROP. XXIII.

If a straight line intercepted between the asymptotes meets the hyperbola, and is bisected in the point where it meets it, it touches the hyperbola; and if it touches the hyperbola, it is bisected in the point of contact.

Fig. 15.
N. 1.

Let there be an hyperbola the asymptotes of which are AB, AC, and let a straight line BC, terminated by the asymptotes, meet it in the point D, and be bisected in D; the straight line BC touches the hyperbola.

Through D draw DE parallel to the one asymptote AC, and meeting the other in E; and in BC take any point G, through which draw GH parallel to DE; GH will meet the hyperbola in [18. 3.] some point F: then, because BD, DC are equal, BE, EA are also equal; and, because of the equiangular triangles, BE is to ED as BH to HG; therefore the rectangle BEA is to the [1. 6. Elem.] rectangle DEA as the rectangle

Fig. 10. N 2

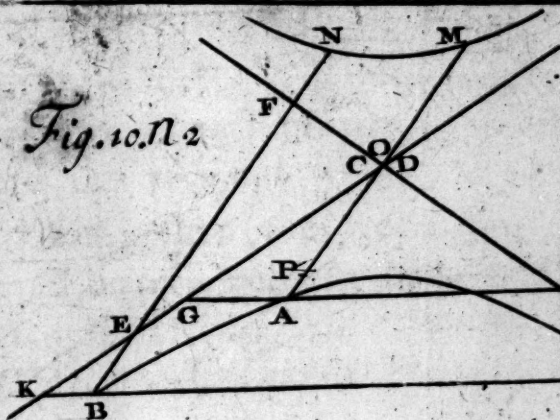


Fig. 11.

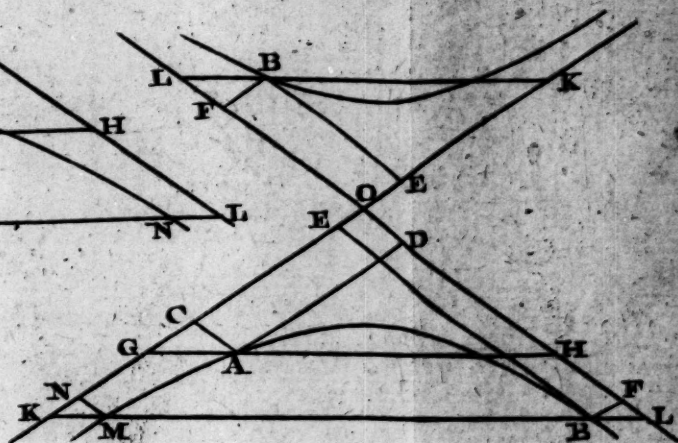


Fig. 12.

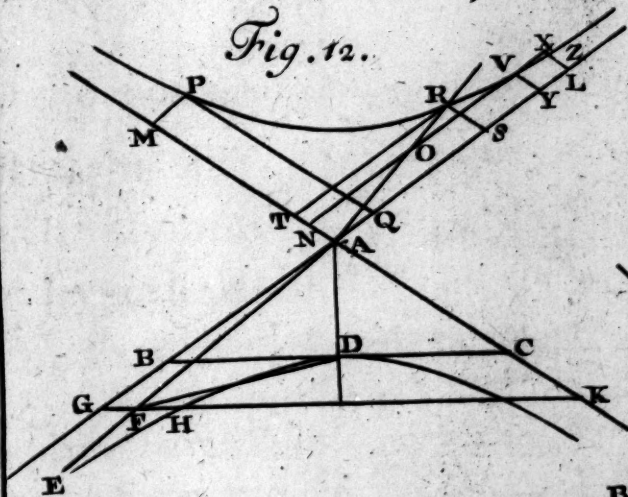


Fig. 13.

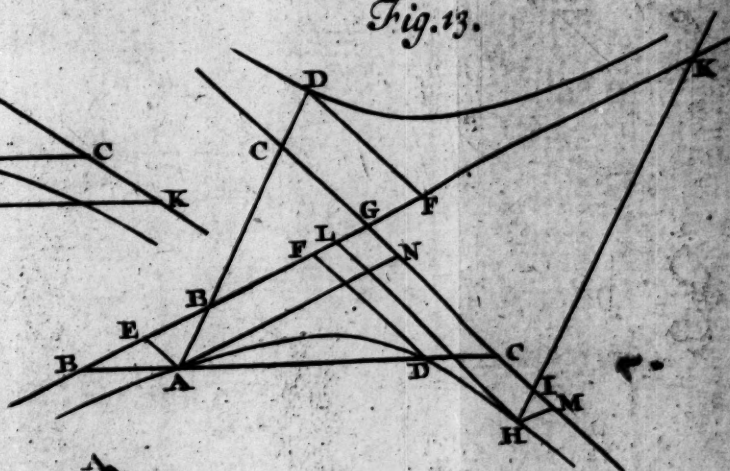
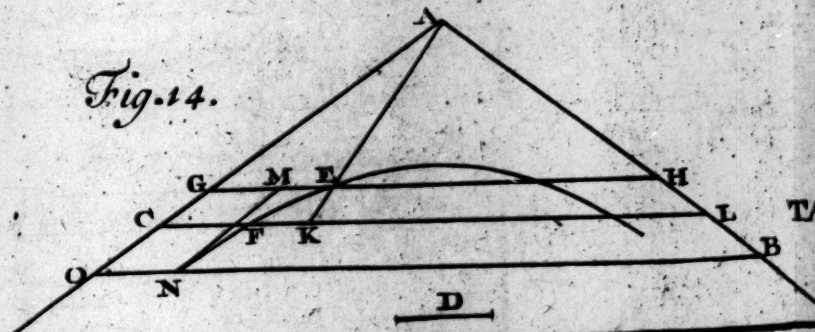
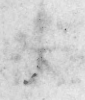


Fig. 14.





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angle BHA to GHA: but the rectangle BEA is [5. 2. Elem.] greater than BHA; therefore the rectangle DEA is greater [14. 5. Elem.] than the rectangle GHA; that is, (F being a point in the hyperbola), the rectangle FHA is greater than the rectangle GHA; and therefore FH is greater than HG; therefore the point G is without the hyperbola; and therefore the straight line BC touches the hyperbola in the point D.

OTHERWISE.

If a straight line LM, terminated by the asymptotes, is bisected by the hyperbola in the point D, it touches the hyperbola.

Fig. 15.
n. 1.

It is plain that LD passes not within the hyperbola; for if it passed within the hyperbola, it would necessarily meet the hyperbola again in another point, because the points L, M are without the hyperbola: but it is impossible for it to meet the hyperbola in any other point but D; for, if possible, let it meet it in N; therefore NM is [14. 3.] equal to DL, that is, according

according to the hypothesis, to DM ; which is absurd: therefore LM falls not within the hyperbola, nor meets it any where but in the point D ; and therefore LM touches it in D . On the contrary, if the straight line LM , terminated by the asymptotes, touch the hyperbola in D , it is bisected in the point of contact.

For if LD , DM are unequal, from DM the greater take away MN equal to LD the less; therefore the point N is [19. 3.] in the hyperbola; and therefore, contrary to the hypothesis, LM cuts the hyperbola.

Fig. 15.
n. 1.

COR. 1. Hence through the same point D of an hyperbola, only one straight line can be drawn touching the hyperbola.

Let D be a point in the hyperbola, and through that point to the asymptote AB draw a straight line DE parallel to the other; and take EB equal to EA , and having joined BD , let it meet the asymptote AC in C : then, since BE , EA are equal, BD , DC are equal; BC , therefore, touches the hyperbola in D . And no other straight line touches it in D : for, if possible, let LDM also touch it; then, since BE , EA are equal, LE , EA are unequal; and consequently

quently LD, DM are unequal: therefore LM does not touch the hyperbola.

COR. 2. Hence a method is pointed out, by which, if the asymptotes AB, AC of an hyperbola be given in position, a straight line BC can be drawn which may touch the hyperbola in a given point D.

COR. 3. If through the vertices of a transverse diameter two straight lines be drawn touching the hyperbolas, they are parallel to each other. Let AC, BC be the asymptotes, and let AOB, QPR touch the hyperbolas in the vertices of the transverse diameter OCP; AB, QR are parallel. There being drawn to either asymptote AC the straight lines OS, PT parallel to the other, the triangles SCO, TCP are equiangular: and since, by the proposition, AO, OB are equal, AS, SC are also equal; and, in like manner, QT, TC are equal: and CO is to ϕ P as CS to CT; and, consequently, as CA to CQ; therefore the triangles OCA, PCQ are equiangular; and therefore OA, PQ are parallel.

COR. 4. And if a straight line be drawn parallel to a tangent, and meeting the hyperbola, the square of the segment (of the tangent) between the point of contact and

A a

either

Fig. 15
n. 2.

either of the asymptotes, is equal to the rectangle contained by the segments of the parallel which are between the asymptotes and the point where it meets the hyperbola; for this rectangle is equal to the rectangle contained by the segments of the tangent [15. 3.] which are between the point of contact and the asymptotes, that is, to the square of the segment (of the tangent) between the point of contact and either of the asymptotes.

PROP. XXIV. PROB.

Fig. 15.
n. 1.

The asymptotes AB, AC of an hyperbola, and a point F in the same, being given in position, to draw a straight line which shall touch the hyperbola, and be parallel to a straight line KO given in position, and which cuts both the asymptotes of the hyperbola, or opposite hyperbolas.

Suppose the problem solved, and let BC
be

be parallel to KO , and touch the hyperbola in D ; and having joined AD , let AD meet KO in P ; draw FRQ parallel to AD , and meeting the asymptotes in Q , R ; and since the straight line BC touches the hyperbola in D , BD is equal to BC [23. 3.]; and consequently KP is equal to PO ; and KO is given in position and magnitude; therefore KP is given, and the point P ; and the point A is given; therefore the straight line PAD is given in position: the square of AD is equal to [1. cor. 15. 3.] the rectangle QFR : and since FRQ is given in position [28. dat.], being drawn through a given point F , parallel to a straight line AD given in position, and that AB , AC are given in position; therefore FQ , FR are [25. 26. dat.] given; therefore the rectangle QFR is given; consequently the square of AD is given; and therefore AD is given in magnitude: but the point A is given in position; therefore the point D is also given [27. dat.]; and therefore the straight line BDC is given in position.

The composition is this: Let KO be bisected in P ; and having joined AP , draw through the point F a straight line FRQ parallel to AP , and meeting the asymptotes

in the points R, Q : again, in AP produced, and in either direction from the centre, let there be placed AD, so that it may be a mean proportional between FQ, FR, and through D draw BDC parallel to KO; BC will touch the hyperbola in D: for since the square of AD is equal to the rectangle QFR, the point D is [cor. 17. 3.] in the hyperbola; and since KO, BC are parallel, and that KO is bisected in P by the straight line PAD, BC is bisected in D; and therefore touches the hyperbola in the same point D.

PROP. XXV. THEOR.

If two straight lines touch an hyperbola, or opposite hyperbolas, and cut the asymptotes; the rectangle contained by the segments (of the asymptotes) intercepted between the centre and the points where the one straight line intersects the asymptotes, is equal to the rectangle contained by the segments (of the asymptotes)

totes) intercepted between the centre and the points where the other straight line intersects the asymptotes.

Let there be an hyperbola, with AB, AD for its asymptotes, and let a straight line BD touch it in C, and a straight line GE touch the same, or the opposite hyperbola, in E; the rectangles BAD, EAG will be equal. Fig. 16.

From the points C, F draw CH, CK, and FL, FM parallel to the asymptotes; then BCD touching the hyperbola, BC is equal to CD, and consequently BA is the double of AH, and AD the double of HC; therefore the rectangle BAD is the quadruple of the rectangle CHA. It may be shewn in the same manner, that the rectangle EAG is the quadruple of the rectangle FMA: but, by proposition 16. of this book, the rectangle CHA is equal to the rectangle FMA; the rectangle BAD is therefore equal to the rectangle EAG.

P R O P.

PROP. XXVI. THEOR.

If two straight lines touching an hyperbola, or opposite hyperbolas, meet the asymptotes; the straight lines drawn between the points of concourse are parallel to each other, and to the straight line joining the points of contact.

Fig. 16. Let there be an hyperbola, with AB, AD for its asymptotes; let BD touch it in C, and EG touch the same, or the opposite hyperbola, in F; BE, DG, when joined, are parallel to CF.

Since the rectangles BAD, EAG are equal, BA is to EA as GA to AD; therefore BE, GD are parallel: join DF, and let it meet BE in N; then since DF is to FN as GF to FE, that is, as [23. 3.] DC to CB, BN, CF are parallel.

COR. Of two straight lines touching an hyperbola, the segments between the asymptotes are cut, proportionally, in the
I
point

point O where the two straight lines intersect each other, and in C, F the points of contact.

PROP. XXVII. THEOR.

Every straight line drawn through the centre of an hyperbola, and passing within the angle adjacent to that within which the hyperbola is described, is a right diameter.

Let there be an hyperbola, AC, BC its asymptotes, and draw any straight line CE through the centre, and within the angle adjacent to the angle ACB; CE is a right diameter.

Fig. 15.
n. 2.

In BC produced take any point D, and through D to CE draw a straight line DF parallel to the asymptote CA; and having made DG equal to DC, join GF, and let GF meet CA in H: then, since GH meets the straight lines CA, CD, which contain the angle adjacent to ACB, it must meet [20. 3.] the hyperbolas: let it meet them in the points K, L; KH, LG are, therefore,

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fore, equal [14. 3.] ; and because CD , DG are equal, and CH , DF parallel, HF , FG are equal; the whole FK is, therefore, equal to the whole FL ; and therefore CF is [4. def. 3.] a right diameter.

D E F. XI.

When a straight line drawn through the centre of an hyperbola, and bisected in the centre, is parallel to a straight line which touches the hyperbola, and equal to the segment (of the tangent) between the asymptotes, it is named the *second diameter* of the diameter drawn through the point of contact.

COR. 1. Hence every second diameter is a right diameter; for it passes within the angle adjacent to that within which the hyperbola is described.

COR. 2. Hence a straight line which joins the vertices of a transverse diameter, and of its second diameter, and thus cuts one of the asymptotes, is parallel to the other asymptote.

For let OCP be a transverse diameter, and MCN its second, and AOB a straight line

line touching the hyperbola in the vertex of the transverse OCP; MO, NO are [31. 1. Elem. and 11. def. 3.] parallel to CB, CA.

D E F. XII.

A third proportional to two diameters, one of which is a transverse diameter, and the other the second diameter of this transverse, is named the *latus rectum*, or the *parameter* of that diameter which is the first of the three proportionals.

P R O P. XXVIII. THEOR.

If from a point in an hyperbola to a transverse diameter a straight line be drawn parallel to the second diameter of that transverse, the square of the transverse diameter is to the square of the second diameter, as the rectangle contained by the segments of the transverse which are between its vertices and the parallel,

B b

l,

lel, is to the square of the parallel.

Fig. 17.
p. 2. 1

Aa being a transverse diameter, and Bb its second diameter, CG , CF the asymptotes, and DE drawn parallel to Bb from a point in the hyperbola to the transverse Aa ; the square of Aa is to the square of Bb as the rectangle AEa is to the square of DE .

Let DE meet the asymptotes in F , G , and draw HAK touching the hyperbola in the vertex A ; therefore, by def. 11. of this book, HA is equal and parallel to BC , and, of consequence, parallel to FE ; and, because of the equiangular triangles, the square of CE is to that of EF as the square of CA to that of AH , that is, as the same square of CA to the [4. cor. 23. 3.] rectangle FDG ; the square of CA is, therefore [19. 5. Elem.], to the square of AH as the rectangle AEa to the square of ED ; the square of Aa , therefore, is to the square of Bb as the rectangle AEa to the square of ED .

COR. 1. The squares of straight lines drawn from points of an hyperbola, or of
↓ the

the opposite hyperbola, to a transverse diameter, and which are parallel to the second diameter of this transverse, are to one another, as the rectangles contained by the segments (of the transverse diameter) intercepted between its vertices and those parallels. See prop. 15. cor. 1. B. 2.

COR. 2. And, on the contrary, A being an hyperbola, having Aa for a transverse diameter, and Bb for the second diameter of Aa ; if from a point D to the transverse Aa a straight line DE be drawn parallel to the second diameter, and meeting the transverse diameter produced in E , and the square of CA be to the square of CB as the rectangle AEa to the square of ED ; the point D is in the hyperbola. For since DE is parallel to BC , and consequently to HK , which touches the hyperbola in the vertex of the transverse diameter, it will necessarily meet the asymptotes [3. cor. 18. 3.], and of consequence the hyperbola: if, then, it does not meet the hyperbola in D , let it, if possible, meet it in another point d , on the same side of Aa with the point D ; therefore the rectangle AEa is to the square of dE as the square of CA to the square of CB , that is, by hypothesis, as

B b 2 the

Fig. 17.

the rectangle AEa to the square of DE ; therefore dE , DE are equal; which is absurd; DE , therefore, meets not the hyperbola in d , nor, as is evident, in any point but D .

COR. 3. Substitute the word *hyperbola* in place of *ellipse*, and the third corollary of prop. 15. B. 2. becomes also a corollary from this proposition.

P R O P. XXIX. THEOR.

If from a point of an hyperbola to a second diameter a straight line be drawn parallel to the transverse of that second diameter; the square of the second diameter is to the square of the transverse diameter, as the sum of the squares of half the second diameter, and of the segment between the centre and the parallel, is to the square of the parallel.

From

From D, a point of the hyperbola, to the second diameter Bb , draw DL parallel to the transverse diameter Aa ; the square of Bb is to the square of Aa as the sum of the squares of CB , CL is to the square of DL . Fig. 17.

Through the point D draw DE parallel to BC ; and since, by the preceding proposition, the square of CA is to the square of CB as the rectangle AEa to the square of ED ; therefore, inversely, and by prop. 12. 5. Elem. the square of CB is to the square of CA as the sum of the squares of CB , ED to the square of CA , together with the rectangle AEa , that is, as the sum of the squares of CB , CL to the square of EC or DL .

COR. 1. If from two points of an hyperbola, or of opposite hyperbolas, to a second diameter, two straight lines be drawn parallel to the transverse of that second diameter, the square of the one straight line is to the square of the other, as the sum of the squares of half the second diameter, and of the distance between the one straight line and the centre, to the sum of the squares of half the same second diameter,

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meter, and of the distance between the other straight line and the centre.

COR. 2. And, on the other hand, if from a point D to a second diameter BC of an hyperbola, a straight line DL be drawn parallel to CA the transverse of that second diameter, and the square of Bb have the same ratio to the square of Aa that the sum of the squares of CB, CL have to the square of DL, the point D is in the hyperbola. For since the straight line DL is parallel to the transverse diameter AC, which is in the space between the asymptotes, it meets the straight lines containing the angle adjacent to that within which the hyperbola is described, and, of consequence, both the hyperbolas [20. 3.] ; and, as in the second corollary of the preceding proposition, it may be proved, that it meets the hyperbola in D.

COR. 3. With the proper alterations, the third corollary of the preceding holds as a corollary here.

PROP.

PROP. XXX. THEOR.

Let there be a transverse diameter, and its second diameter; any straight line terminated both ways by the hyperbola, or the opposite hyperbolas, and parallel to either of these diameters, is bisected by the other of them; or, what is the same thing, a transverse diameter, and its second, are conjugate diameters.

COR. It is evident, that two diameters cannot be conjugated to the same diameter, whether it be a transverse or a second diameter.

PROP. XXXI. THEOR.

Let there be a transverse diameter, and its second diameter; any straight line terminated both ways by an hyperbola, or opposite
site

site hyperbolas, and bisected by either of these diameters, is parallel to the other of them: and therefore straight lines ordinate-ly applied to either of these diameters are parallel to each other.

COR. 1. There being a transverse, and its second diameter, straight lines parallel to either of them, and which cut off equal segments of the other, between the points where they meet it and the centre, are equal; and if parallel to either diameter, and equal to each other, they cut off, between the centre and the points where they meet the other diameter, equal segments of this other diameter.

Fig. 17.
n. 2.

These two propositions, and this first corollary, are demonstrated from the 28th and 29th propositions, in the same manner in which the 9th and 10th propositions were demonstrated from the 7th and 8th.

COR. 2. If two or more parallels are terminated both ways by an hyperbola, or hyperbolas, the diameter which bisects the
one,

one, or one of them, bisects the other, or the rest of them; for that parallel which is bisected, is parallel to the diameter conjugated to the bisecting diameter; the other, or the rest, therefore, is, or are, parallel to the same conjugated diameter, and therefore is or are bisected by the other diameter [30. 3.].

COR. 3. A straight line, on the other hand, which bisects two parallels terminated both ways by an hyperbola, or opposite hyperbolas, is a diameter: for if not, draw a diameter bisecting the one of the parallels; this diameter will bisect the other of them; but, by hypothesis, there is another straight line which bisects both; which is absurd.

COR. 4. If a straight line touch an hyperbola, a straight line drawn through the point of contact, and which bisects a straight line terminated both ways by the hyperbola, and parallel to the touching line, is a diameter; for a parallel to the tangent is [11. def. 3.] parallel to the diameter conjugated to that which passes through the point of contact: and if the straight line drawn through the point of contact, and which bisects the straight line

C c

mentioned,

mentioned, be not a diameter, draw a diameter through the point of contact; and this diameter will also bisect [30. 3.] the parallel to the touching line, or to the conjugated diameter; which is absurd.

COR. 5. Two straight lines terminated both ways by an hyperbola, or by opposite hyperbolas, and not passing through the centre, do not bisect each other: for if they are both terminated by the same hyperbola, or by opposite hyperbolas, draw a diameter through the point where they intersect each other; and then, by the proposition, they will be both parallel to the diameter conjugated to this diameter; which is absurd: but if the one of them be terminated by the hyperbola, and the other drawn between the opposite hyperbolas, it is evident that they cannot bisect each other.

P R O P. XXXII. THEOR.

A straight line drawn through the vertex of a transverse diameter, and which is parallel to a straight line ordinately applied to that diameter,

diameter, touches the hyperbola; and if it touch the hyperbola, it is parallel to straight lines ordinately applied to the transverse diameter drawn through the point of contact.

Let there be an hyperbola, the asymptotes of which are CF , CG ; let Aa be a transverse diameter, and through the vertex A of Aa draw HAK parallel to DM ordinately applied to the same Aa ; HK touches the hyperbola. Fig. 17.

The straight line ordinately applied to the transverse diameter Aa is parallel [31. 3.] to the second diameter conjugated to that transverse; therefore HAK drawn through the vertex A of Aa , is parallel to the same conjugated second diameter; and therefore it touches the hyperbola [11. def. 3.]. And, on the other hand, if HK touch the hyperbola, it is parallel to the second diameter [11. def. 3.] of CA : but the ordinate DM is parallel to the same [31. 3.]; therefore HK , DM are parallel to each other.

PROP. XXXIII. THEOR.

A straight line that touches an hyperbola meeting a diameter, there being drawn from the point of contact a straight line ordinately applied to that diameter; the semidiameter is a mean proportional between its segments intercepted between the centre and the ordinate, and between the centre and the tangent.

Case 1. When the touching line meets a transverse diameter.

Fig. 18.

There being an hyperbola, its asymptotes AG, AH, a straight line KCG being drawn touching that hyperbola in the point C, and meeting a transverse diameter BAO in E, and CD drawn from the point of contact C, so as to be ordinately applied to BAO; AD, AB, AE are proportionals.

Through the vertex B let GBF be drawn parallel

parallel to CD, and let it meet in the point N the tangent drawn through C, and let BM drawn parallel to HC meet AG in M, and let DC meet AH in L, and join KF, GH; then, because GBF, drawn through the vertex of the diameter, is parallel to the ordinate DC, it [32. 3.] touches the hyperbola; and since the tangents HK, GF are cut proportionally in [cor. 26. 5.] the points C, B, N; CN is to NH as BN to NG; and therefore, because of the parallels, LF is to FH as MK to KG; and since KF, GH are [26. 3.] parallel, FH is to FA as KG to KA; therefore, *ex aequo*, LF is to FA as MK to KA: and, by composition, LA is to FA as MA to KA; therefore, because of the parallels, DA is to BA as BA to EA.

Case 2. When the tangent meets a second diameter of a transverse diameter.

The straight line CE touching the hyperbola, and meeting both the second diameter AB in E, and KAF (which is conjugated to AB) in G, and CD, CH being drawn from the point of contact C, so as to be ordinate-ly applied to the conjugate diameters; AD, AB, AE are proportionals.

For by the preceding case, AH, AF, AG are proportionals; therefore the square of AH

Fig. 19.

AH is to the square of AF as AH is [con. 20. 6.] to AG; and, by division, the rectangle KHF is to the square of AF as HG to GA: but since CH is ordinately applied to AF, the rectangle KHF is to the square of AF as the square of CH or AD to the square of AB; therefore the square of AD is to the square of AB as HG to GA; that is, as CH or AD to AE; and therefore AD, AB, AE are proportionals [convers. cor. 20. 6. Elem.]

Fig. 18.

COR. 1. Since (in the case where the tangent and ordinate meet the transverse BAO) AD, AB, AE are proportionals, DO is to DB as EO to EB, that is, the segments (of the diameter) between its vertices and the ordinate are to each other as the segments of the same between the tangent and the same vertices. The demonstration is similar to that given in the second part of prop. 17. B. 2.

COR. 2. When (in the second case) the ordinate drawn from the point of contact passes through the extremity of the second diameter, the tangent passes through the other extremity of the same diameter: for since the distance between the ordinate and

and the centre is equal to the half of the second diameter, the distance between the tangent and the centre is equal to the same semidiameter.

PROP. XXXIV. THEOR.

If from a point C in an hyperbola a straight line CD be ordinately applied to the diameter AB , and a straight line CE be drawn from the same point; if the semidiameter be a mean proportional between the segments of AB , which are cut off towards the centre by these straight lines; CE touches the hyperbola.

For if CE does not touch the hyperbola, let CP touch it; therefore, by the preceding proposition, AD, AB, AP are proportionals: but, by the hypothesis, AD, AB, AE are proportionals; which is absurd: CE , therefore, touches the hyperbola.

Fig. 18.
19.

PROP.

PROP. XXXV. THEOR.

If a straight line touching an hyperbola meet a transverse diameter, there being drawn from the point of contact a straight line ordinately applied to the same diameter; the rectangle contained by the segments (of the diameter) intercepted between the ordinate and the centre, and between the ordinate and the tangent, is equal to the rectangle contained by the segments between the ordinate and the vertices of the diameter; and the rectangle contained by the segments between the tangent and the centre, and between the tangent and the ordinate, is equal to the rectangle contained by the segments between the tangent and the vertices

vertices of the diameter. But if the tangent meet a second diameter, there being drawn from the point of contact a straight line ordinately applied to that second diameter; the rectangle contained by the segments between the ordinate and the centre, and between the ordinate and the tangent, is equal to the sum of the squares of the semidiameter and the segment between the ordinate and the centre; and the rectangle contained by the segments between the tangent and the centre, and between the tangent and the ordinate, is equal to the sum of the squares of the semidiameter and the segment between the tangent and the centre.

Case 1. The straight line which touches the hyperbola in C, meeting the transverse
D d diameter

Fig. 18.

diameter BAO in the point E, and an ordinate meeting the same diameter in D; the rectangle ADE is equal to the rectangle BDO, and the rectangle AED to BEO.

For since AD, AB, AE are proportionals, the rectangle DAE is equal to the square of AB; and these equals being taken from the square of AD, the remaining rectangle ADE is equal to the remaining rectangle BDO [2. and 6. 2. Elem.]. Next, from the same equals, viz. the rectangle DAE and the square of AB, take away the common square [3. and 5. 2. Elem.] of AE, and the remaining rectangle AED is equal to the remaining rectangle BEO.

Fig. 19.

Case 2. The tangent and ordinate drawn from the point C, meeting the second diameter in the points E, D; since the rectangle EAD is equal to the square of AB, add to each of these equals the square of AD, and the rectangle ADE will be equal to [3. 2. Elem.] the sum of the squares of AB, AD. Next, if to the same equals, to wit, the rectangle EAD and the square of AB, the square of AE be added; the rectangle AED will be equal to the sum of the squares of AB, AE.

P R O P.

PROP. XXXVI. THEOR.

If a straight line touch an hyperbola it bisects the angle contained by the straight lines drawn from the foci to the point of contact; and, on the contrary, if a straight line bisect the angle contained by two straight lines drawn from a point of an hyperbola to the foci, it touches the hyperbola.

There being an hyperbola, its transverse axis AB, and the centre the point C, a straight line DE touching it, and meeting the transverse axis in E, and the straight lines DF, DG being drawn from D the point of contact to the foci; the angles FDE, GDE are equal.

Fig. 29.

From the point D draw DH perpendicular to the axis, and from the point A, which is the nearer (to D) of its vertices, place, in the axis produced, a straight line AK equal to DF, and KB will [1. 3.] be

D d 2 equal

equal to DG; and, by the fifth proposition of this book, CK is to CH as CF to CA: but as CH to CA, so [33. 3.] is CA to CE; therefore, *ex equali*, as CK to CA so is CF to CE; and, by conversion, as CK is to KA so is CF to FE; and by doubling the antecedents, twice CK is to KA as FG to FE: therefore, by division, BK is to KA as GE to FE; and BK, KA are equal to DG, DF; therefore as DG to DF, so is GE to EF; and therefore the straight line DE bisects the angle FDG [3. 6. Elem.].

If, on the contrary, a straight line DE bisects the angle FDG, it touches the hyperbola: for if not, let another straight line touch the hyperbola in the point D; this other straight line will bisect the angle FDG, which, by the hypothesis, is bisected by DE; which is absurd. The demonstration here might have been similar to the second demonstration in prop. 11. B. 2.

PROP.

Fig. 15. N. 1.

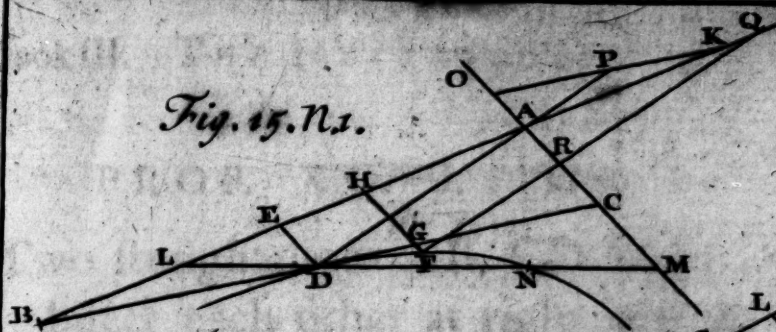


Fig. 16.

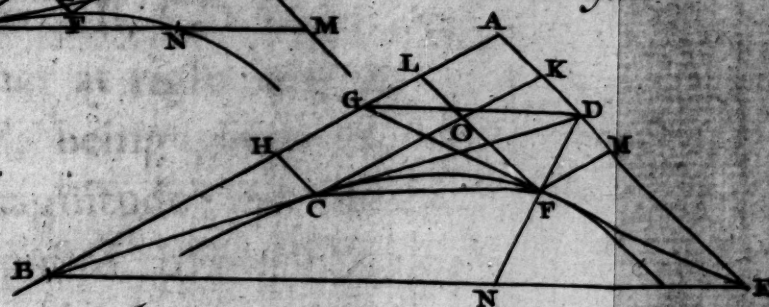


Fig. 15. N. 2.

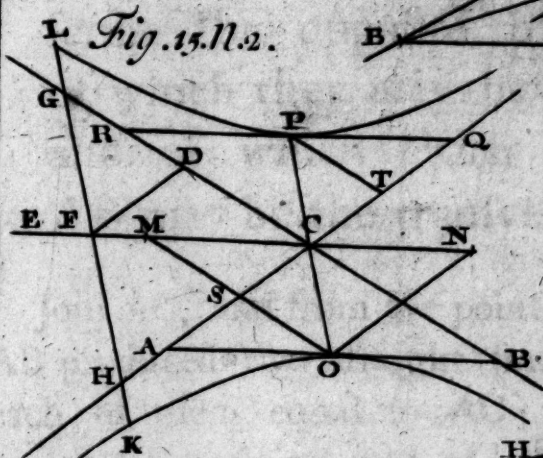


Fig. 17. N. 1.

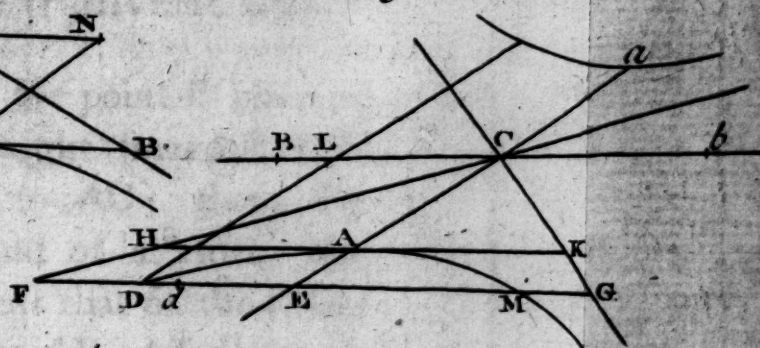


Fig. 17. N. 2.

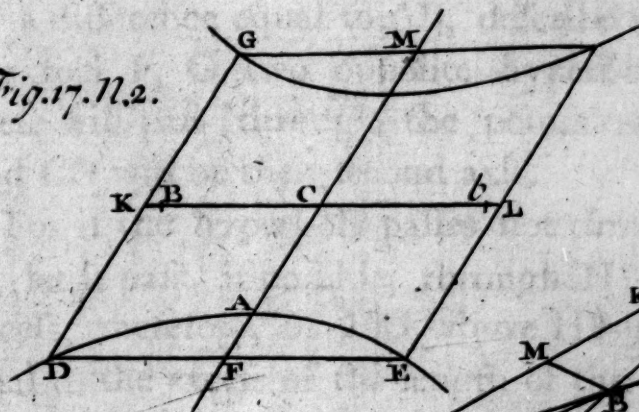
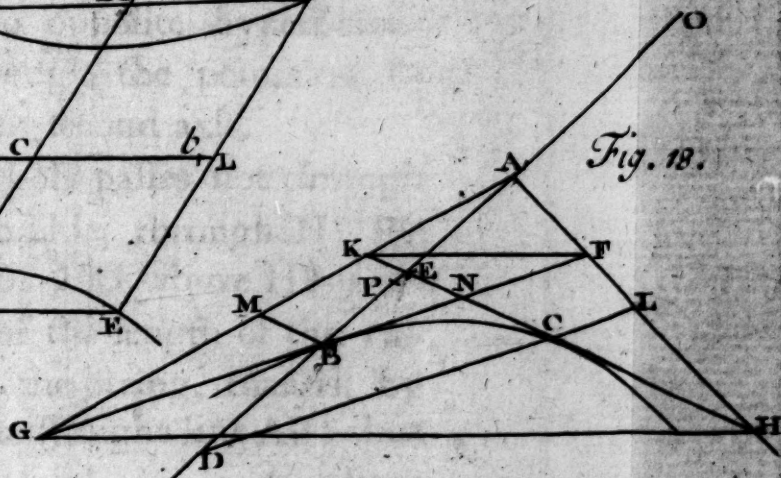
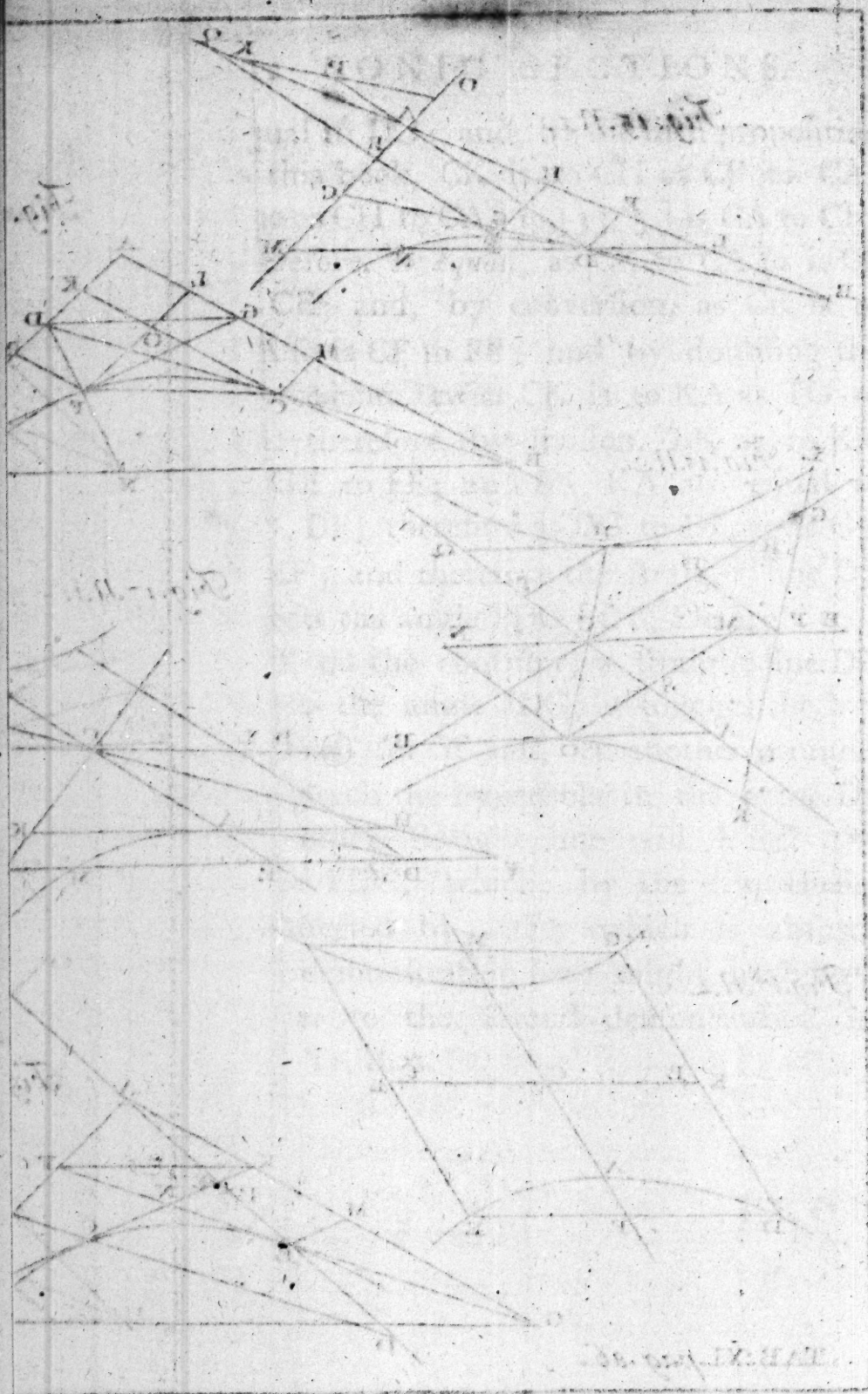


Fig. 18.





PROP. XXXVII. PROB.

Two straight lines AB, CD which Fig. 21.
 bisect each other at right angles
 in the point E, being given in
 position and magnitude; to de-
 scribe the opposite hyperbolas
 of which they may be the axes,
 and of which either of them
 AB may be the transverse axis.

Join AC, and from the point E place in
 AB produced two straight lines EF, EG,
 each of them equal to AC; then, by
 means of a string and of a ruler, the
 length of which exceeds that of the string
 by a difference equal to AB, describe with
 the foci F, G two opposite hyperbolas;
 these will pass through the points A, B,
 and CD will be their second axis.

For if the hyperbola passes not through
 A, let it pass, if possible, through H; the
 excess, therefore, of HG above HF is e-
 qual to the excess of the length of the ru-
 ler above that of the string, that is, by
 construction, to the straight line AB: but
 since

since BG is equal to AF, the excess of AG above AF is equal to the same AB; which is absurd: the hyperbola, therefore, passes through A: and in like manner it may be shewn that it passes through B. Again, C, D are the extremities of the second axis: for if C be not one of its extremities, let the point K, on the same side of the centre on which C is, be one of them; therefore KA being joined will be equal [6. def. 3.] to EF; and, by construction, CA is equal to the same EF; therefore KA is equal to CA: which is absurd.

D E F. XIII.

Fig. 22.

If upon two straight lines Aa, Bb, which bisect each other at right angles, two opposite hyperbolas AG, ag be described, and upon the same straight lines other two opposite hyperbolas BK, bk be described, so that Bb, the transverse axis of the latter hyperbolas, may be the second axis of the former, and that Aa, the second axis of the latter, may be the transverse axis of the former; these four are named *conjugate hyperbolas*.

P R O P.

PROP. XXXVIII. THEOR.

The asymptotes of conjugate hyperbolas are common to the four hyperbolas.

Let there be conjugate hyperbolas the axes of which are Aa , Bb , and let the straight lines CD , CE be the asymptotes of two opposite hyperbolas, the transverse axis of which is Aa ; the same straight lines are the asymptotes of the two opposite hyperbolas, the transverse axis of which is Bb . Fig. 22.

Through the vertex A draw DAE parallel to BC , and join DB , and produce it to F ; therefore, by the 10th definition of this book, BC is equal and parallel to AD or AE ; BD is therefore equal and parallel to CA ; and because the triangles FBC , CAE are equal and equiangular, BF is equal to CA ; therefore CD , CF are [10. def.] asymptotes of the hyperbola, the transverse axis of which is Bb , and the second axis Aa .

PROP.

PROP. XXXIX. THEOR.

If from a point G in one of four conjugate hyperbolas, a straight line GH be drawn parallel to EC , one of the asymptotes, and meeting the other in H , there being drawn from the point K in the adjacent hyperbola a straight line KL parallel to either asymptote, and meeting the remaining one in L ; the rectangles GHC , KLC contained by the parallels, and the segments of the asymptotes between the parallels and the centre, are equal. On the contrary, if the point G be in one of the conjugate hyperbolas, and the point K within the angle contained by the asymptotes of the adjacent hyperbola, the rectangle KLC being at the same time

time equal to the rectangle GHC ; the point K is in the adjacent hyperbola.

Let Aa , Bb be the axes of conjugate hyperbolas; join AB , and let it meet the asymptote CD in M , and draw AD parallel to CB ; then, since BC , AD are equal and parallel, the triangles CBM , ADM are similar and equal; and consequently AM , MB are equal: the rectangles AMC , BMC are therefore equal, and AB is parallel to the [2. cor. def. 10. 3.] asymptote EC ; therefore the rectangles GHC , KLC are equal to the rectangles AMC , BMC ; and consequently they are equal to each other.

Fig. 221

On the contrary, if G be a point in one of the conjugate hyperbolas, and the point K be within the angle DCF , and the rectangle KLC equal to the rectangle GHC , with the same construction in other respects still remaining; the point K is in the adjacent hyperbola: for since the rectangle KLG is equal to GHC , that is, to AMC , that is, to BMC , and that B is in the ad-

E c jacent

adjacent hyperbola; the point K is [4. cor. 18. 3.] in the same hyperbola.

Fig. 23.

COR. 1. If a straight line DE, intercepted between an hyperbola and one of the adjacent hyperbolas, is bisected by one of the asymptotes, it is parallel to the other: for let DE meet the asymptote CG in L, and to the other asymptote draw DH, EK parallel to CG; then, since DL, LE are equal, and that DH, LC, EK are parallel, HC, CK are equal; and by the proposition, the rectangles DHC, EKC are equal; DH, EK are therefore equal, and they are parallel; therefore DE, KH are parallel.

COR. 2. If DE be parallel to the asymptote KH; DL, LE are equal: for by the proposition, the rectangles DLC, ELC are equal.

COR. 3. If, lastly, DE be parallel to the asymptote HK, and DL, LE being equal, the point D be in one of the hyperbolas; the point E is in the adjacent hyperbola: for since DL, LE are equal, the rectangles DLC, ELC are equal; therefore, by the proposition, E is in the adjacent hyperbola BE.

P R O P.

PROP. XL. [*Prop. 20. B. 2. Apoll.*]

If a straight line touch one of four conjugate hyperbolas, and through their centre two straight lines be drawn, the one meeting the hyperbola in the point of contact, and the other parallel to the tangent, and meeting one of the adjacent hyperbolas; this other straight line drawn parallel to the tangent, is the half of the second diameter conjugated to the transverse diameter drawn through the point of contact: and, on the contrary, if half a second diameter be drawn conjugated to the transverse diameter drawn through the point of contact, its extremity is in the adjacent hyperbola.

Let there be conjugate hyperbolas the asymptotes of which are CG, CF, and let

Fig. 23.

E e 2

MF

MF touch one of the hyperbolas in D; join CD, and draw CE parallel to MF, and meeting in the point E the hyperbola adjacent to that in which D is; CE is the half of a second diameter conjugated to CD.

Through the points D, E draw the straight lines DH, EK parallel to the asymptote CG, and meeting the other asymptote in H, K: then, since the straight line MF touches the hyperbola in D, MD, DF are equal; therefore CH, HF are equal; therefore the rectangle DHF is equal to the rectangle DHC, that is, by the preceding proposition, to the rectangle EKC: and since the triangles EKC, DHF are equiangular, EK is to DH as KC to HF; therefore the square of EK is to the square of DH, as the rectangle EKC to the rectangle DHF: but the rectangles EKC, DHF, as has been proved, are equal; therefore the squares of EK, DH are equal; and therefore EK, DH are likewise equal; consequently EC, DF are [26. 1. Elem.] equal, and they are parallel; therefore CE is half [11. def. 3.] the second diameter conjugated to CD.

If, on the contrary, CE (the same construction remaining) be half the second
diameter

diameter conjugated to CD , its extremity E is in the hyperbola adjacent to that where the point D is : for CE is equal and [11. def. 3.] parallel to DF , and EK is parallel to DH ; consequently the triangles EKC , DHF are [26. 1. Elem.] equal; KC , consequently, is equal to HF or HC , and EK to DH : and for this reason, the rectangle EKC is equal to the rectangle DHC , and the point D is in the hyperbola, and the point E is within the angle adjacent to that within which this hyperbola is described; the point E , therefore, is in the adjacent hyperbola (by part 2. of the preceding.).

Fig. 23.

COR. 1. If CD be half a transverse diameter, and CE half the second diameter conjugated to CD in the hyperbola AD ; CE is a transverse, and CD a second diameter conjugated to CE in the adjacent hyperbola BE .

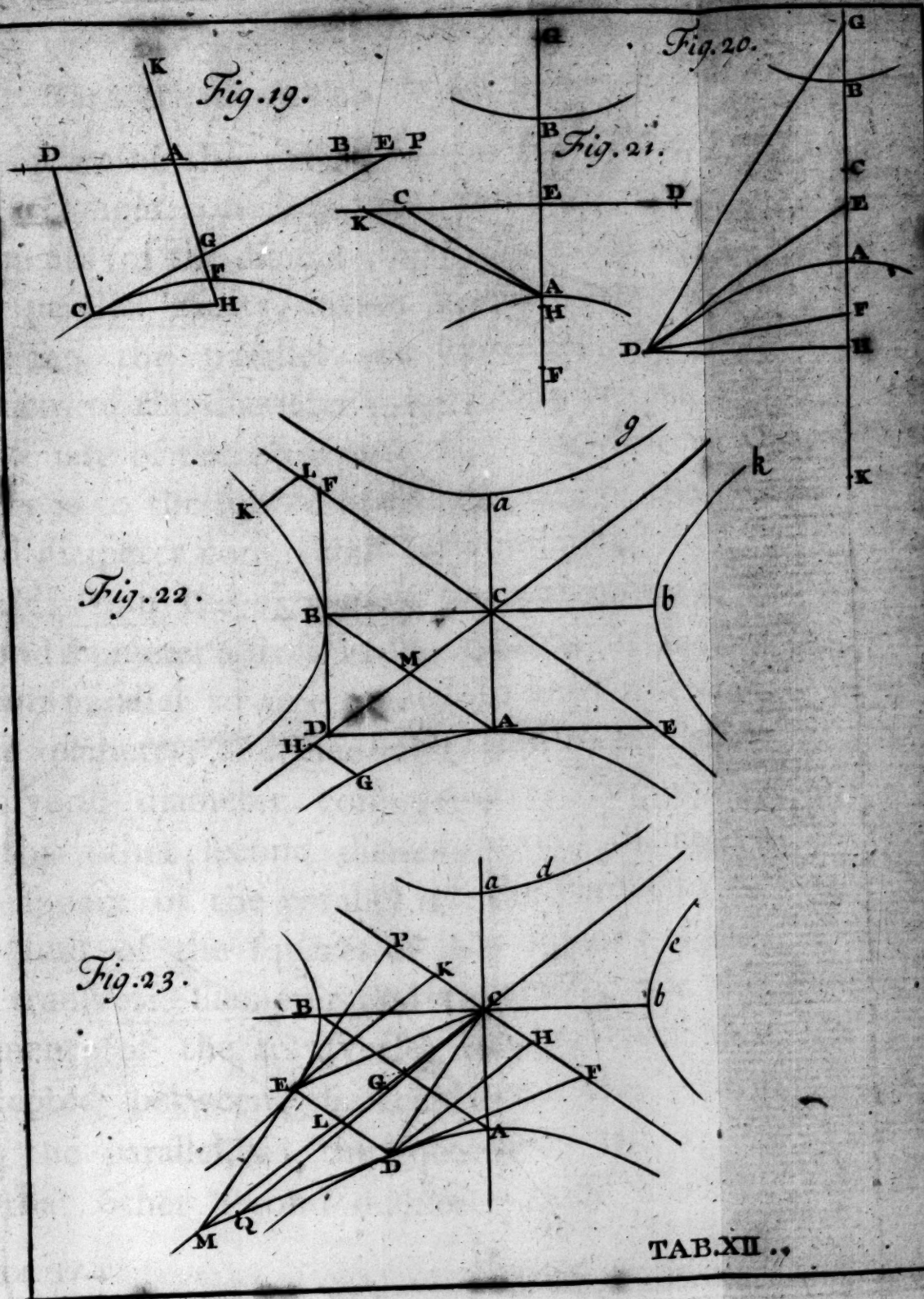
DE and ME being joined, let ME meet the other asymptote CF in P ; then, since DE is [2. cor. def. 11. 3.] parallel to CF , and that MF is bisected in D , MP is bisected in E , and the point E is in the adjacent hyperbola; therefore MP touches that hyperbola:

perbola: and CD is equal and parallel to ME ; for CE , MD are equal and parallel; therefore CD is the second diameter conjugated to the transverse diameter CE in the [11. def.] hyperbola BE .

COR. 2. The same construction still remaining, the straight line which joins the centre C , and the point of concurrence M of the tangents DM , EM , drawn through the vertices of the conjugate diameters, is an asymptote: for if CM be not an asymptote, let CQ , meeting DM in Q , be one; CE , therefore, is equal to DQ : but the same CE is equal to DM ; because $DMEC$ is a [11. def.] parallelogram; DQ and DM are therefore equal; which is absurd.

PROP. XLI. THEOR.

If from the extremity of a second diameter of an hyperbola a straight line be drawn parallel to any transverse diameter, and meeting the second diameter conjugated to this transverse; the



of the parallel
 angle contained by
 (of the diameter wh
 the meets) intercept
 the parallel and
 at the diameter met
 e of the transverse
 to the square of the
 not conjugated to
 from the extremity
 a straight line
 parallel to any other
 meter, and meeting
 diameter conjugate
 other second diamet
 re of the parallel is
 of the squares of
 verse diameter and
 (of the transverse)
 between the cen
 parallel, as the squ
 other second diamet



the square of the parallel is to the rectangle contained by the segments (of the diameter which the parallel meets) intercepted between the parallel and the vertices of the diameter met, as the square of the transverse diameter is to the square of the second diameter conjugated to it: And if from the extremity of a second diameter a straight line be drawn parallel to any other second diameter, and meeting the transverse diameter conjugated to this other second diameter; the square of the parallel is to the sum of the squares of half the transverse diameter and the segment (of the transverse) intercepted between the centre and the parallel, as the square of that other second diameter is

is to the square of the transverse diameter conjugated to it.

Fig. 24.

In the first case, let there be an hyperbola, the transverse diameter of which is DCd , and let $KCKk$ be the second diameter conjugated to DCd , and let CB be any other second diameter, and from its extremity B draw a straight line parallel to the transverse diameter CD , and meeting in L the second diameter conjugated to CD ; the square of BL is to the rectangle KLk as the square of Dd to the square of Kk .

- For the points B, K, k , which are the extremities or vertices of the second diameters, are in the adjacent hyperbolas; the transverse diameter of which Kk is conjugated to the second [1. cor. 40. 3.] diameter Dd ; therefore the square of BL is to the rectangle KLk as the square of Dd to the square of Kk [28. 3.].

As to the second case, it is demonstrated in the very same manner from the 29th proposition of this book.

COR. If from any point A of an hyperbola AD a straight line AM be drawn ordinately applied to the right diameter Kk ,
and

and from B, an extremity of any second diameter CB to the same Kk, a straight line BL be drawn parallel to AM; the square of BL, is to the square of AM, as the rectangle KLk is to the sum of the squares of the semidiameter KC and the segment CM between the centre and the ordinate. This is evident from the proposition, and from the 29th of this book.

But if from any point A of an hyperbola AD a straight line AE be drawn ordinately applied to the transverse diameter Dd, and from the extremity B of the second diameter to the same Dd a straight line BF be drawn parallel to AE; the square of BF is to the square of AE, as the sum of the squares of the semidiameter CD, and the segment CF, between the centre and BF, is to the rectangle DEd. This is demonstrated from the proposition, and from the 28th of this book.

¶

PROP.

PROP. XLII. THEOR.

Fig. 24

If from the extremity B of a second diameter CB a straight line BF be drawn parallel to straight lines ordinately applied to any diameter Dd, and BH be drawn parallel to the diameter CA, which is conjugated to CB, and meeting the diameter Dd in H; the semidiameter CD is a mean proportional between CF and CH.

For the extremity B of the second diameter is in the adjacent hyperbola, and CA is a [1. cor. 40. 3.] second diameter of that adjacent hyperbola; therefore BH touches the same [11. def.]; and therefore CF, CD, CH are [33. 3.] proportionals.

Cor. The 35th proposition is equally true when accommodated to this case.

PROP.

PROP. XLIII. THEOR.

If from the extremities of two conjugate diameters of an hyperbola straight lines be drawn ordinarily applied to any third transverse diameter; the square of the segment (of the third diameter) intercepted between the centre and the ordinate drawn from the extremity of the transverse diameter, is equal to the square of the segment of the same third diameter between the centre and the ordinate drawn from the extremity of the other of the conjugate diameters, together with the square of half the third diameter; and the square of the segment (of the third diameter) intercepted between the centre and the ordinate drawn from the extremi-

ty of the second diameter, is equal to the rectangle contained by the segments (of the third diameter) between the ordinate drawn from the extremity of the other of the conjugate diameters and the vertices of the same third diameter.

Fig. 34.

Let there be opposite hyperbolas, in which CA is the half of a transverse diameter, and let CB be the second diameter conjugate to CA , and let Dd be any other transverse diameter, and from the extremities A, B draw AE, BF ordinately applied to Dd ; the square of EC is equal to the square of FC together with the square of CD ; and the square of FC is equal to the rectangle DEd .

To the diameter Dd draw AG parallel to BC , and BH parallel to AC ; therefore, because of the parallels, the triangles CAG, HBC are equiangular; and since AE, BF are parallel, CAE, HBF are also equiangular; CE, HF consequently, is to HF as CA to BH ; that is, as CG to CH : and since

CD

CD is a mean proportional both between CE and CG, and between CF and CH [33. and 42. 3.], CF is to CE as CG to CH; and therefore CF is to CE as CE to HF; consequently the square of CE is equal to the rectangle CFH: but the rectangle CFH is equal to the sum of the squares of CF, CD; consequently the square of CE is equal to the sum of the same squares of CF, CD: take the square of CD from each of these equals, and there will remain the rectangle DEd equal to the square of CF.

COR. Hence the semidiameter CD, to which the ordinates are drawn, is to the conjugate semidiameter CK, as the distance between the one ordinate and the centre is to the remaining ordinate: for the square of CD is to the square of CK as the rectangle DEd is to the square of EA, that is, according to the proposition, as the square of CF is to the square of EA; and therefore CD is to CK as CE to EA. Again, because the square of CD is to the square of CK as the sum of the squares of CF, CD to the square of BF, that is, by the proposition, as the square of CE to the square of BF; CD is to CK as CE to BF.

P R O P.

PROP. XLIV. THEOR.

The excess of the squares of any conjugate semidiameters is equal to the excess of the squares of the halves of the axes, if the conjugate diameters be unequal: and if any one diameter be equal to its conjugate, any other diameter is also equal to its conjugate; and in this case, the angle contained by the asymptotes is a right angle.

Fig. 24.

Let CA, CB be conjugate semidiameters, and CD, CK the halves of the axes, and from A, B draw the straight lines AE, AM and BF, BL ordinates to the axes; therefore the excess of the squares of CA, CB is equal to the excess, by which the sum of the squares of CE, EA differs from the sum of the squares of CL, LB: but, by the preceding, the square of CE is equal to the sum of the squares of CF, CD; and by the same proposition, the square of CL is equal
to.

to the sum of the squares of CM, CK; therefore the excess of the squares of CA, CB is equal to the excess by which the sum of the squares of CF, CD, EA differs from the sum of the squares of CM, CK, LB; and the squares of CF, LB are equal; as also the squares of EA, CM: therefore, if these equals be taken away, the excess by which the sum of the squares of CF, CD, EA differs from the sum of the squares of CM, CK, LB, is equal to the excess by which the square of CD differs from the square of CK; and therefore the excess of the squares of CA, CB is equal to this same excess.

Otherwise: Let AB, AC be the halves Fig. 25. of any two transverse diameters in an hyperbola, AD, AE the asymptotes; and draw the straight lines BD, CE touching it in the points B, C, and meeting the asymptotes in D, E; therefore, by the 11th def. and prop. 30. of this book, BD is equal to half the second diameter conjugated to AB; and CE, in like manner, is equal to half the second diameter conjugate to AC: it is to be proved, that the excess of the squares of AB, BD is equal to the excess of the squares of AC, CE.

Through

Through the points B, C, draw BF, CH parallel to the asymptotes, and BG, CK perpendicular to them: therefore the rectangles AFB, AHC are equal [1. cor. 16. 3.]; and, of consequence, AF is to AH as HC to FB, that is, since the triangles are equiangular, as HK to FG; consequently the rectangles AFG, AHK are equal, and their quadruples are equal: and since, through the point of contact B, a straight line BF is drawn parallel to the asymptote AE; DF, FA are equal; consequently DG is equal to AF, together with FG: and hence four [8. 2. Elem.] times the rectangle AFG is equal to the excess of the squares of DG, GA, that is, since the triangles DGB, AGB are right-angled, to the excess of the squares of DB, BA. It may in the same manner be shewn, that four times the rectangle AHK is equal to the excess of the squares of EC, CA; and four times the rectangle AFG, as hath been proved, is equal to four times the rectangle AHK; consequently the excess of the squares of DB, BA is equal to the excess of the squares of EC, CA.

But if in an hyperbola any transverse diameter AB is equal to the second diame-

ter BD conjugated to it, any other transverse diameter in the same hyperbola is also equal to its conjugate second diameter, and the angle contained by the asymptotes is a right angle; for since DB, BA, and DF, FA are equal, and BF common, in the triangles DBF, ABF; the angle BFD, and of consequence the angle EAD, is a right angle; and since EAD is a right angle, the angle CHE is also a right angle; and EH, HA are equal, and CH common; therefore EC, CA are equal.

PROP. XLV. THEOR.

If through the vertices of two conjugate diameters four straight lines be drawn touching conjugate hyperbolas, the parallelogram formed by them is equal to that formed by the tangents drawn through the vertices of any other two conjugate diameters.

Let AB, CD be conjugate diameters, and Fig. 26.

G g through

through their vertices draw tangents, meeting each other in K, L, M, N ; and let EF, GH be any other conjugate diameters, and through the vertices of these draw tangents, meeting each other in O, P, Q, R ; the figures $KLMN, OPQR$ are parallelograms, and equal.

Let S be the centre of the hyperbola; and since both KN, LM , and KL, MN are [3. cor. 23. 3.] parallel, the figure $KLMN$ is a parallelogram. For a like reason, $OPQR$ is a parallelogram: and since AK, CK touch the hyperbolas in the vertices of conjugate diameters, the point K where they meet is in an asymptote; and the rest of the angles, as may be proved in like manner, of the parallelograms, have their vertices in the asymptotes: therefore the asymptotes are the diagonals of the parallelograms; consequently the parallelogram $KLMN$ is the quadruple of the triangle KSN , and the parallelogram $OPQR$ the quadruple of the triangle OSR : but the triangles KSN, OSR are equal, because the rectangles KSN, OSR are equal [25th of this book, and 15. 6. Elem.]; therefore the parallelograms $KLMN, OPQR$ are also equal.

qual. This proposition might also have been demonstrated like prop. 20. B. 2.

P R O P. XLVI. THEOR.

If two conjugate diameters of an hyperbola meet a straight line touching the hyperbola, the rectangle contained by the segments (of the tangent) intercepted between the point of contact and the conjugate diameters, is equal to the square of the semidiameter conjugated to that diameter which passes through the point of contact.

Let ACB, DCE be two conjugate diameters, and let a straight line which touches the hyperbola in F meet them in the points G, H, and let CK be the semidiameter conjugated to CF; the rectangle GFH is equal to the square of CK.

Fig. 274

From the points F, K draw to AB the straight lines FM, KL parallel to DE: then, because of the parallels, GM is to MC as

G g 2

GF

GF to FH; consequently the rectangles GMC, GFH are similar: and because the triangles GMF, CLK are equiangular, GM is to GF as CL to CK; therefore the rectangles GMC, GFH, and the squares of CL, CK, which are four similar rectilinear figures, are proportionals, being similarly described upon the four proportional straight lines GM, GF, CL, CK: but the rectangle GMC is equal to [35. 3.] the rectangle AMB, that is, to the [43. 3.] square of CL; therefore the rectangle GFH is equal to the square of CK.

PROP. XLVII. THEOR.

If from a point of an hyperbola a straight line be drawn ordinate-ly applied to a transverse diameter, the rectangle contained by the segments (of the diameter) intercepted between its vertices and the ordinate, is to the square of the segment (of the ordinate) intercepted between the hyperbola and the diameter,

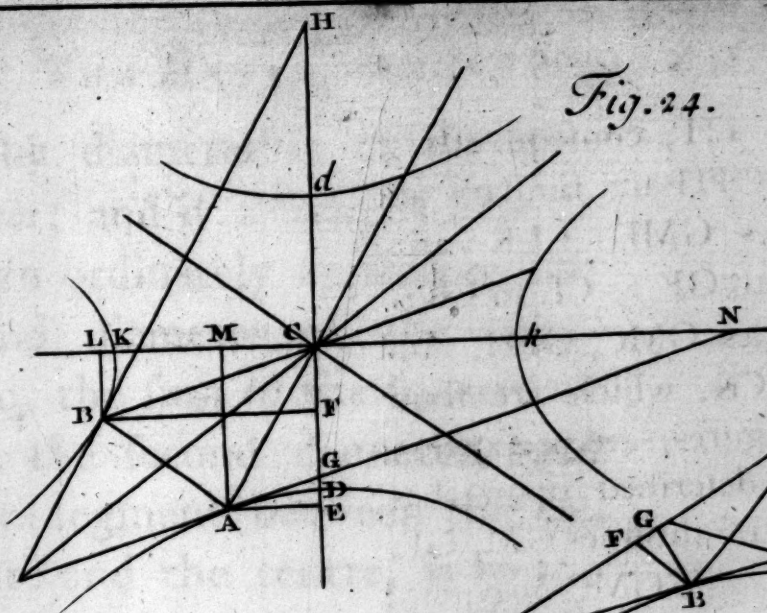


Fig. 24.

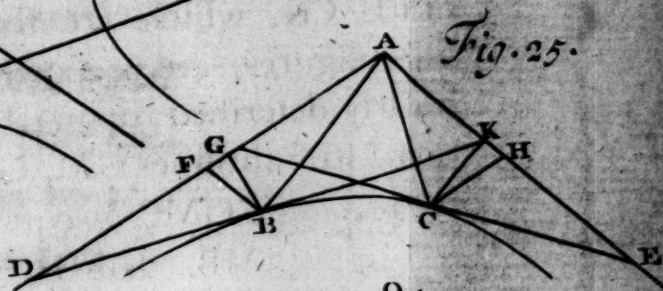


Fig. 25.

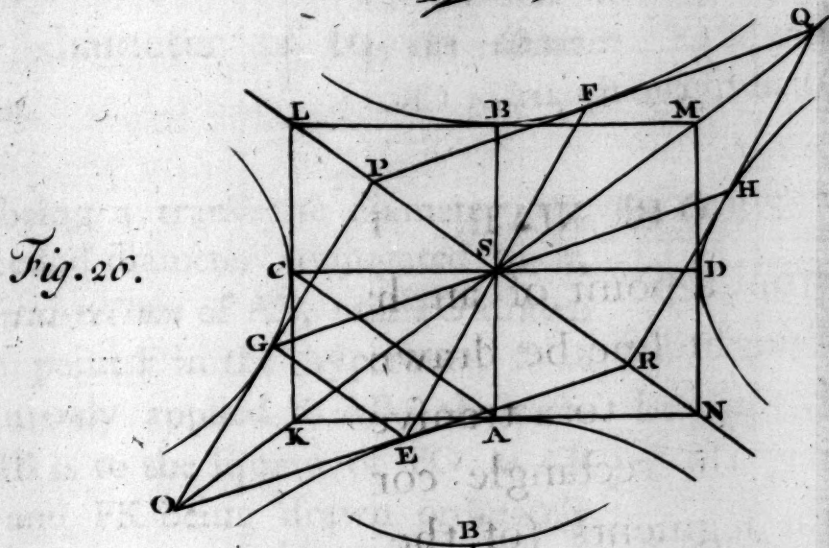


Fig. 26.

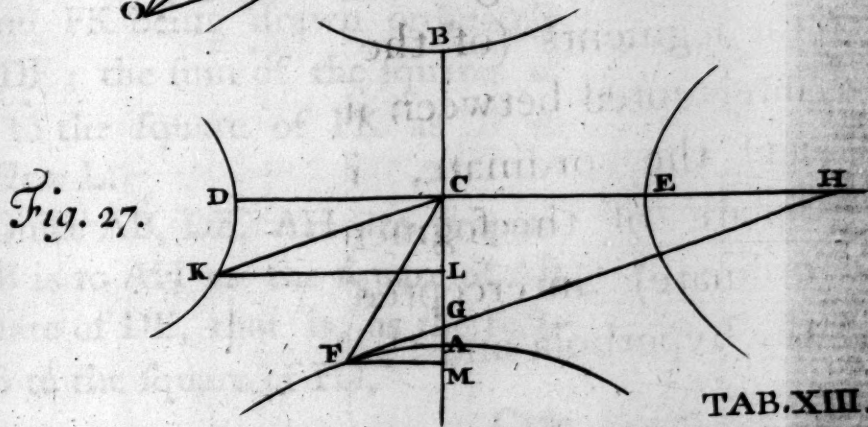


Fig. 27.

Fig. 14

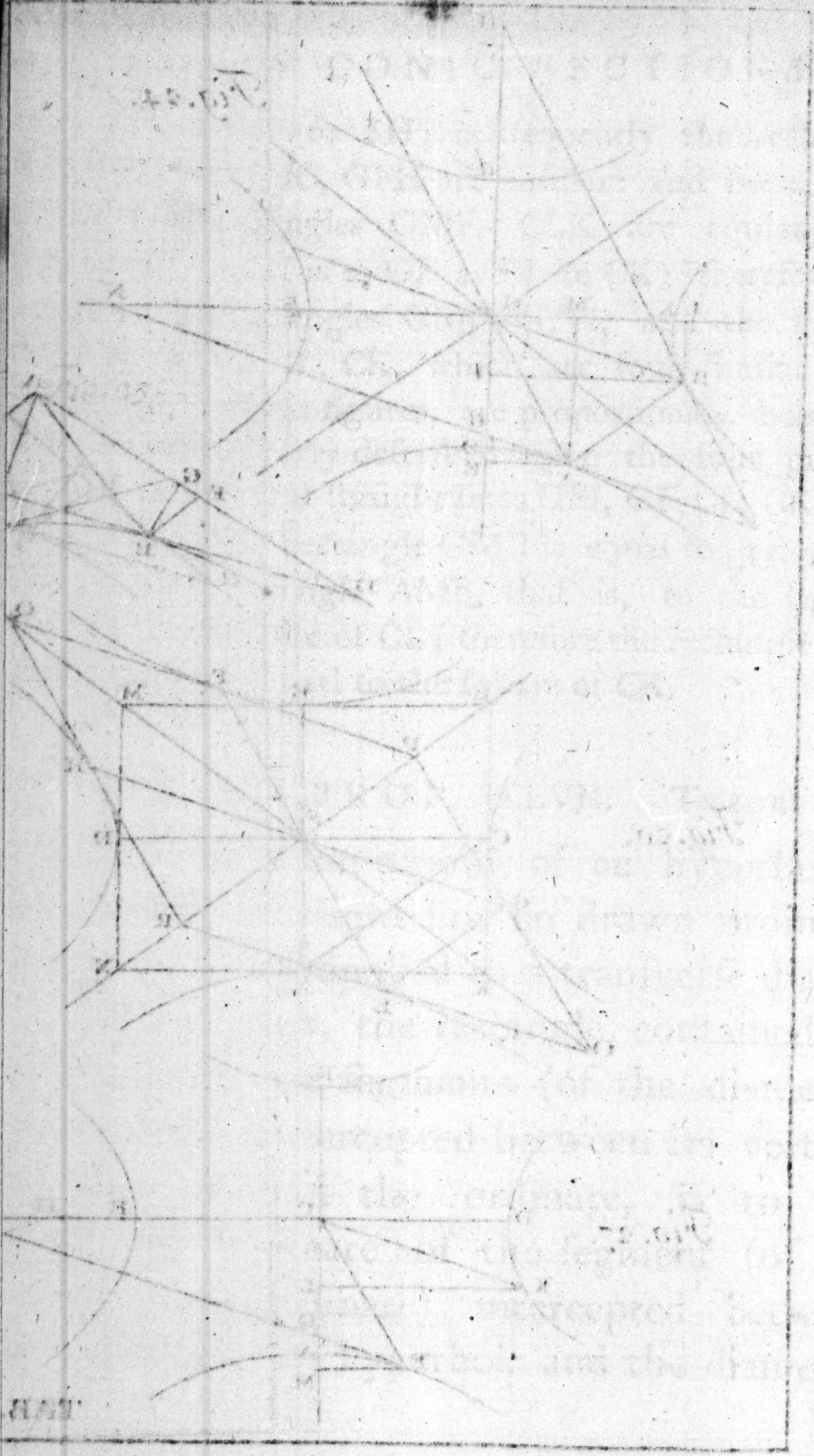


Fig. 14

as the diameter is to its *latus rectum*; and if a straight line be drawn ordinately applied to the second diameter of the transverse, the sum of the squares of half the second diameter, and of its segment between the ordinate and the centre, is to the square of the ordinate, as the second diameter is to its *latus rectum*.

There being a transverse diameter AB, DE the second diameter conjugated to it, AH the *latus rectum* of AB, and FG drawn from the point F in the hyperbola, so as to be ordinately applied to AB; the rectangle AGB is to the square of FG as AB to AH: and FK being drawn ordinately applied to DE; the sum of the squares of CD, CK is to the square of FK as DE to its *latus rectum* L.

Case 1. Since AB, DE, AH are proportionals, AB is to AH as the square of AB to the square of DE, that is, as the rectangle AGB to the square of FG.

Case

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Case 2. And since DE , AB , L are proportionals, DE is to L as the square of DE is to the square of AB , that is, as the sum of the squares of CD , CK is to the square of KF .

PROP. XLVIII. THEOR.

Fig. 28. If from a point F in an hyperbola a straight line FG be ordinately applied to a transverse diameter AB , and from the vertex of that diameter a straight line AH be drawn perpendicular to AB , and equal to its *latus rectum*; the square of the ordinate is equal to the rectangle applied to the *latus rectum*; which rectangle has for its breadth the abscissa between the ordinate and the vertex, and exceeds by a figure similar, and similarly situated, to that which is contained by the diameter and the *latus rectum*.

BH

BH being joined, GM drawn from the point G parallel to AH, and meeting BH in M, and MN drawn through the point M parallel to AB, and meeting AH in N; the rectangles MNHO, BAHP are completed.

Then since the rectangle AGB is to the square of FG, as AB to AH, that is, as GB to GM, that is, as the rectangle AGB to the rectangle AGM; AGB is to the square of GF as the same AGB to the rectangle AGM; therefore the square of GF is equal to the rectangle AGM; which having the abscissa AG for its breadth, is applied to the *latus rectum* AH, and exceeds the rectangle HAGO by the rectangle MNHO similar to BAHP. From that square's being equal to the exceeding rectangle, Apollonius named the curve line which is the subject of this third book, the *hyperbola*.

COR. It is evident, that the square of GF would be equal to the rectangle AGM, though AH were not at right angles to AB.

P R O P.

PROP. XLIX. PROB.

Fig. 29.

A straight line AB being given in position and magnitude, and a point F being given; to describe an hyperbola, of which AB may be the transverse axis, and which may pass through the point F; but the given point must be so situated, that a perpendicular drawn from it towards AB may fall upon AB produced.

Draw FG at right angles to AB, and find a straight line DE such, that the square of AB may be to that of DE as the rectangle AGB to the square of FG: let AB, DE bisect each other at right angles; then, with AB, DE as axes, and making the former the transverse axis, describe an {37. 3.} hyperbola AF; this hyperbola will pass [2. cor. 28. 3.] through the point F.

PROP. L. PROB.

A straight line DE being given in position and magnitude, and a point F being given; to describe an hyperbola, of which DE may be the second axis, and which may pass through F. Fig. 29.

Bisect DE in C, and draw FH perpendicular to DE, and find a straight line AB such, that the square of DE may be to that of AB as the sum of the squares of CD, CH to the square of FH; let DE, AB bisect each other at right angles; then with the axes AB, DE, and making the former the transverse axis, describe [37. 3.] an hyperbola; this hyperbola will pass through the point F [2. cor. 29. 3.].

H h

PROP.

PROP. LI.

Of all transverse diameters in an hyperbola the transverse axis is the least; and the angle contained by any other transverse diameter, and a tangent drawn through its vertex, is less than a right angle.

Fig. 29.

Let there be an hyperbola, CA the half of its transverse axis, and CF the half of any other diameter; from F, the vertex of CF, draw FG perpendicular to the axis CA; therefore CF is greater than CG; and consequently much greater than CA: Draw a straight line touching the hyperbola in the point F, and meeting the axis CA in K; and since the angle CFG is acute, CFK must be still more acute.

PROP.

PROP. LII. PROB.

Of an hyperbola AF given in position, to find a diameter, the centre, the axes, the asymptotes, and the foci.

Draw two parallel straight lines, and let them be terminated both ways by the hyperbola; and the straight line which bisects them is a [3. cor. 31. 3.] diameter; and any other diameter may be found in the same manner; and the point where two diameters thus found meet each other is the [4. def. 3.] centre. But if two opposite hyperbolas be given in position, the point which bisects the diameter first found is the centre.

Take in the hyperbola any point F, and from the centre C draw CF, and from the centre C, with the distance CF, describe a circle; if this circle meet the hyperbola no where but in the point F, CF is the least of the transverse diameters, and is, consequently, the transverse axis: but if the circle meet the hyperbola again in another point L, join FL, and let it be bisected

Fig. 29.

sected in the point G ; join also CG , and let it meet the hyperbola in A ; CA is half the transverse axis: For since FG , GL are equal, FL is ordinately applied to the diameter CG ; and consequently a straight line which is drawn through the vertex A parallel to FL touches the hyperbola [32. 3.]; and the angle contained by this tangent, and the diameter CA , is a right angle, for the angle FGA is a right angle; therefore, by the preceding proposition, CA is the transverse axis.

Next, in order to find the second axis, draw through the centre C a straight line at right angles to CA , and in that straight line take CD , and let the square of CA have the same ratio to that of CD which the rectangle BGA has (CB being made equal to CA) to the square of GF ; CD will be the second axis, as is evident from prop. 7. of this book.

Lastly, having found the axes, find the asymptotes from def. 10.

But if two opposite hyperbolas be given in position, the asymptotes may be found more easily in this manner. Draw through the centre C any transverse diameter AB ; draw likewise a straight line parallel to AB ,
and

and terminated in the hyperbolas in the points O, P ; and to the straight line OP apply, on both sides, a rectangle equal to the square of CA , and deficient by a square; which is possible, since CA is less than the half of OP [4. cor. 15. 3.]; and let Q, R be the points of application; CQ, CR , when joined, will be the asymptotes [3. cor. 15. 3.]. The foci are found in the manner mentioned in the 57th proposition,

PROP. LIII. PROB.

The asymptotes of an hyperbola being given in position, and a point in it being given; to find the axes of the hyperbola, and to describe it.

Let AC, BC be the asymptotes given in position, and D be the given point. Suppose the problem solved, to wit, let FCE, GCH be the axes, the former of which, as it is within the angle ACB , in which the point D is, must be the transverse axis: Draw through E a straight line parallel

Fig. 30.

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rallel to GH, and let this parallel meet
 the asymptotes in the points K, L; conse-
 quently KL is equal [10. def. 3.] to GH,
 and is bisected in E: and since in the tri-
 angles KEC, LEC, KE, EL are equal, and
 the angles at E right angles; the angles
 ECK, ECL are equal: but the angle KCL
 is given; consequently its half KCE is gi-
 ven: and KC, and the point C, are given
 in position; consequently CE is given
 [29. dat.] in position: through the given
 point D let DMN be drawn parallel to CE,
 and let it meet the asymptotes in M, N;
 DN is therefore given in [28. dat.] posi-
 tion, and the points M, N [25. dat.];
 therefore DM, DN are given in magnitude
 [26. dat.]; the rectangle MDN is conse-
 quently given in magnitude: but the
 square of CE is equal to this rectangle
 [1. cor. 15. 3.]; the square of CE is, there-
 fore, given in magnitude; and conse-
 quently CE is given [55. dat.] in magni-
 tude; but, as hath been proved, it is also
 given in position; the point E, therefore,
 and the straight line KEL, are [27. 29.
 dat.] given in position; and consequently
 KEL is given in magnitude, because CA,
 CB are given in position: and GH is pa-
 rallel

parallel and equal to KL ; consequently GH is given in magnitude; but it is also given in position, because C is given, which bisects it; the axes, therefore, EF , GH are given in position and magnitude; therefore the hyperbola may be described by prop. 37. of this book.

The composition is as follows. Let the angle ACB be bisected by the straight line CE ; and having drawn DMN parallel to CE , make the squares of CE , CF each of them equal to the rectangle MDN ; and through E draw KEL perpendicular to CE , and meeting the straight lines AC , CB in the points K , L ; and through C let GCH be drawn equal and parallel to KEL , so that at the same time it may be bisected in C : then, with the axes EF , GH , and making the former of them the transverse axis, describe an hyperbola; AC , BC will be its asymptotes, and it will pass through the point D : for since, by construction, KEL is equal and parallel to the second axis GH , and is bisected in E ; CK , CL are [def. 10. 3.] the asymptotes, and the rectangle MDN is equal to the square of CE ; the point D , therefore, is in the hyperbola.

And

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And the asymptotes AC , BC being given, and a point D of an hyperbola, as many points of that hyperbola, or of the opposite hyperbola, as may be thought necessary, may be found, by drawing through D any number of straight lines ADB , Dab , meeting the asymptotes in A , B , and a , b ; and taking BO , bo equal to AD , aD , in such a manner, that the two points D , O , and the two D , o , may be either both within or both without the points A , B , and a , b ; for the points O , o will be [19. 3.] in the hyperbolas.

PROP. LIV. PROB.

Fig. 31.

Two straight lines ACB , DCE , which bisect each other in C , being given in position and magnitude; to describe two opposite hyperbolas, which may have AB for a transverse diameter, and DE for the second diameter conjugated to AB .

Suppose what is required done, and let CF , CG be the asymptotes; through A , the

the vertex of the transverse diameter, draw a straight line parallel to DE , and let it meet the asymptotes in F, G ; FG , therefore, is [11. def. 3.] equal to DE , and is bisected in A : but DE is given in magnitude; therefore FG is given in magnitude; and of consequence its half AF is also given in magnitude: but AF is given [28. dat.] in position, since it is drawn through a given point A parallel to DE given in position; consequently the point F is [27. dat.] given: In like manner the point G is given; and the point C is given; therefore the asymptotes CF, CG are given in position, and the point A is given in the hyperbola; which may, therefore, be described by the preceding proposition.

The composition is as follows. Through the vertex A of the transverse diameter draw a straight line FAG equal and parallel to the second diameter DE , and in such a manner that it may be bisected in A ; join CF, CG , and, by the preceding proposition, describe an hyperbola, which may have for its asymptotes the straight lines CF, CG , and which may pass through the point A : and, in the same manner, by employing the point B , describe the oppo-

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finite hyperbola; AB will be a transverse diameter in these hyperbolas, and DE the second diameter conjugated to it.

For since CF, CG are the asymptotes of the hyperbolas, and that through the point A, in the one of the hyperbolas, there is drawn a straight line FAG, which is bisected in A; FG touches [23. 3.] the hyperbola in A: and DE is equal and parallel to FG, and is bisected in the centre C; therefore DE is the second diameter conjugated [11. def. and 30. 3.] to the transverse diameter AB.

PROP. LV. PROB.

The position and magnitude of a diameter of an hyperbola being given, and the position being given of a straight line which is ordinately applied to that diameter from a given point of the hyperbola; to describe the hyperbola.

Case 1. When the given diameter is a transverse diameter of the hyperbola.

Let

Let AB be the given transverse diameter, to which a straight line HK, given in position, is ordinately applied from a given point H of the hyperbola; and let AB be bisected in C, and through C draw a straight line parallel to HK; and in that straight line take CD and CE equal to each other, so that the rectangle AKB may be to the square of HK as the square of AC to the square of CD or CE; and, by the preceding proposition, describe two opposite hyperbolas, of which AB may be a transverse diameter, and DE the second diameter conjugated to AB; one of these hyperbolas will pass through the point H [2. cor. 28. of this book.].

Fig. 31.

Case 2. When the given diameter is a second diameter.

Let DE be the given second diameter, to which HL given in position, and drawn from H, a given point in the hyperbola, is ordinately applied; and let DE be bisected in C, and through C draw a straight line parallel to HL, and in that straight line take CA and CB equal to each other, so that the sum of the squares of LC, DC may be to the square of HL as the square of DC to the square of AC or CB; and, by

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the preceding proposition, describe two opposite hyperbolas, having AB for a transverse diameter, and CD for the second diameter conjugate to AB: of these hyperbolas, the one which lies on the same side of DE with the point H, will pass through the point H [2. cor. 29. 3.].

PROP. LVI. THEOR.

If a cone cut by a plane through the axis, be cut likewise by a second plane, cutting its base in the direction of a straight line perpendicular to the base of the triangle through the axis; and the common section of the triangle through the axis, and the second plane, meet, without the vertex of the cone, a side of the triangle through the axis; the line which is the common section of the second plane, and the conical surface, is

is an hyperbola, having for a transverse diameter the common section of the triangle through the axis and the second plane.

Let there be a cone, its vertex the point **A**, and base the circle **BC**; let it be cut by a plane through the axis, and let the triangle **ABC** be the section; let it be cut also by another plane, cutting its base in the direction of the straight line **DE** perpendicular to **BC**, the base of the triangle **ABC**; and let the section made in the surface of the cone be the line **DFE**; and let the straight line **FG**, the common section of the triangle through the axis, and the second plane, be produced, and without the vertex of the cone, and in the point **H**, let it meet the side **AC** of the triangle **ABC**; the line **DFE** is an hyperbola, which has **FG** for one of its transverse diameters.

For in the section **DFE** take any point **K**, and through **K** to **FG** draw **KL** parallel to **DE**, and through **L** draw **MN** parallel

Fig. 32.

parallel to BC ; the plane, therefore, which passes through KL , MN is [15. 11. Elem.], parallel to the plane through DE , BC , that is, to the base of the cone; and therefore the [23. 1.] plane through KL , MN is a circle, of which MN is a diameter: but KL is [10. 11. Elem.] perpendicular to MN , because DE is perpendicular to BC ; the rectangle, therefore, MLN is equal to the [35. 3. Elem.] square of KL : and in like manner, the rectangle BGC is equal to the square of DG ; the square of DG is therefore to the square of KL as the rectangle BGC to the rectangle MLN : but BG is to ML as FG to FL ; and GC is to LN as GH to LH ; therefore the ratios compounded of these ratios are the same to one another; and therefore the rectangle BGC is to [23. 6. Elem.] the rectangle MLN as the rectangle FGH to the rectangle FLH : hence, in like manner, the square of DG is to the square of KL as the rectangle FGH to the rectangle FLH : Describe, therefore, an hyperbola [55. 3.], of which FH may be a transverse diameter, and in which DG may be ordinately applied to FH : and

Fig. 28.

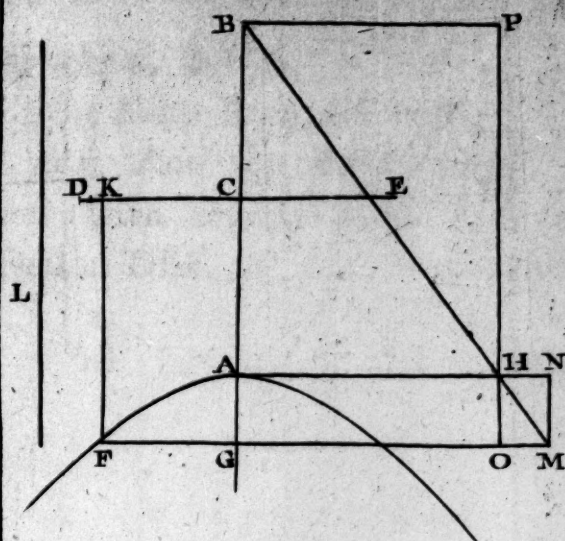


Fig. 29.

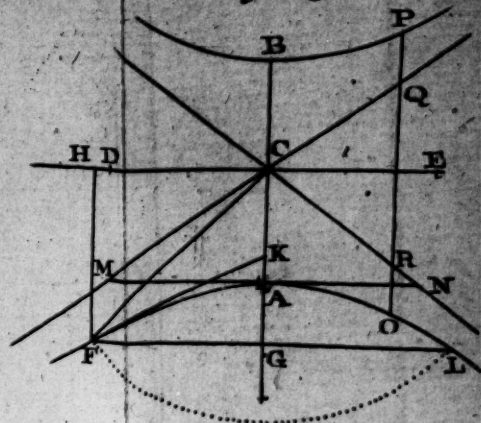


Fig. 30.

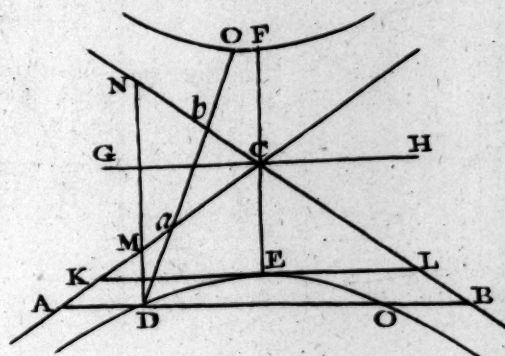


Fig. 31.

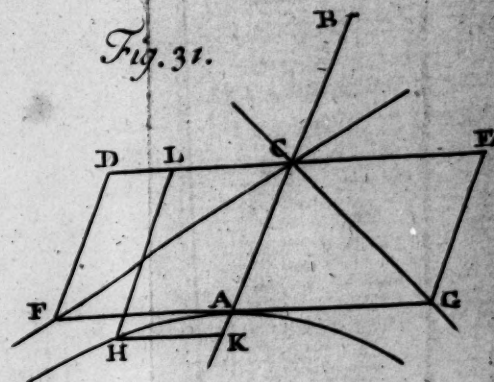
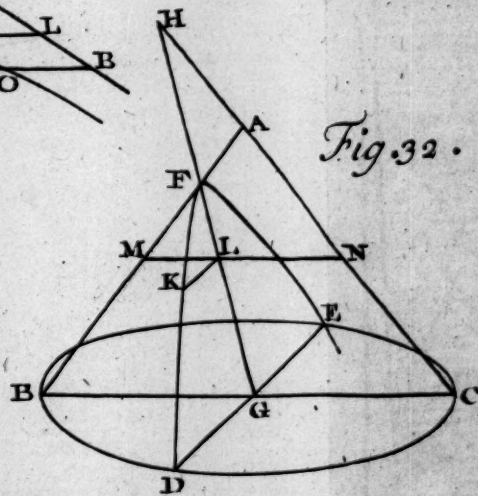
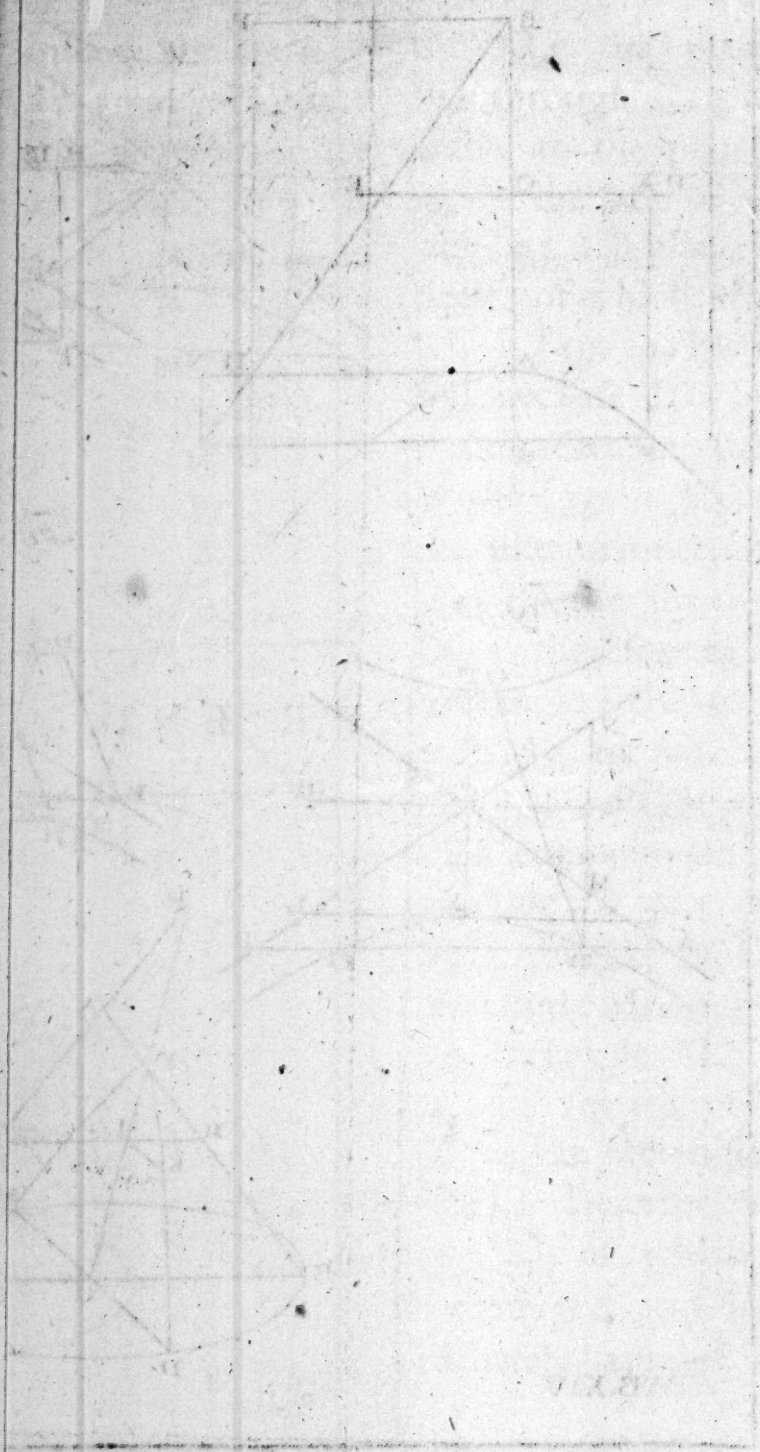


Fig. 32.



19.11



since, by construction, the point D is in this hyperbola, the point K is likewise in it [3. cor. 28. 3.]. And the same thing may be proved with regard to all the points of the section DEF.

THE END.

Book III. THE HYPERBOLAE.

Since, by construction, the point D is in
this hyperbola, and likewise in
it [i.e. the other hyperbola] the same thing
may be proved to all the
points of the [i.e. the other hyperbola]



THE END